



**Autonomous Vehicle Simulation (AVS) Laboratory,
University of Colorado**

**Basilisk Technical Memorandum
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SPHERICAL PENDULUM DYNAMICS MODEL**

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Scope/Contents
<p>The Basilisk sphericalPendulum model module is responsible for simulating an oscillating spherical pendulum in spacecraft tanks. This phenomenon makes the fuel exchange energy and momentum with the spacecraft, thus creates disturbances on attitude and on precise orbit determination. This model together with the linearSpringMassDamper, that is a spring mass damper model, provides an efficient and precise simulation of the real phenomenon.</p>

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1 Model Description

1.1 Problem Statement

The formulation assumes that there is a rigid hub, with N_P lumped masses in the tank for the fuel. Subscript j is used to indicate the j^{th} fuel slosh mass, m_j . Figure 1 displays the frame and variable definitions used for this formulation.

There are four coordinate frames defined for this formulation. The inertial reference frame is indicated by $\mathcal{N} : \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$. The body fixed coordinate frame, $\mathcal{B} : \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$, which is anchored to the hub and can be oriented in any direction. The initial pendulum frame, $\mathcal{P}_{0,j} : \{\hat{p}_{0,j,1}, \hat{p}_{0,j,2}, \hat{p}_{0,j,3}\}$, is a frame with its origin located at tank geometrical center, T . The $\mathcal{P}_{0,j}$ frame is a fixed frame respect to the body frame, oriented such that $\hat{p}_{0,j,1}$ points to the fuel slosh mass in its initial position, P_j . The constant distance from point T to point P_j is defined as l_j .

There are a few more key locations that need to be defined. Point B is the origin of the body frame, and can have any location with respect to the hub. Point B_c is the location of the center of mass of

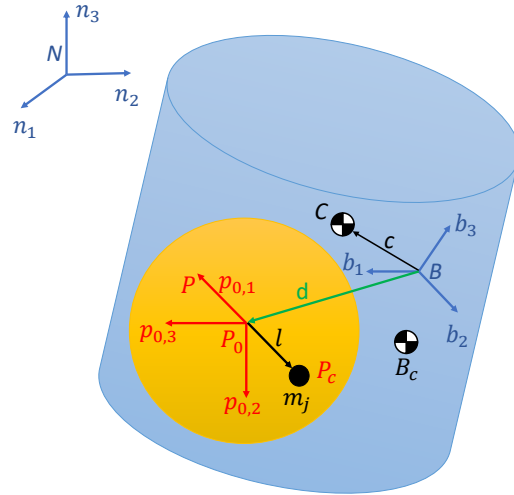


Fig. 1: Frame and variable definitions used for formulation

the rigid hub. $P_{c,j}$ is the instantaneous position of the fuel slosh mass m_j . \mathbf{d} is vector from the center of the body reference system to the tank geometrical center. \mathbf{l}_j is the vector from T to $P_{c,j}$.

Figure 2 provides further detail of the fuel slosh parameters and reference frames. As seen in Figure 1, an individual slosh particle is free to move in every direction while connected by rigid weightless rod to the geometrical center of the tank. A linear damper effect is considered using a damping matrix, D . The variables, φ_j and ϑ_j are state variables and quantify the angular displacement from initial position for the corresponding slosh mass.

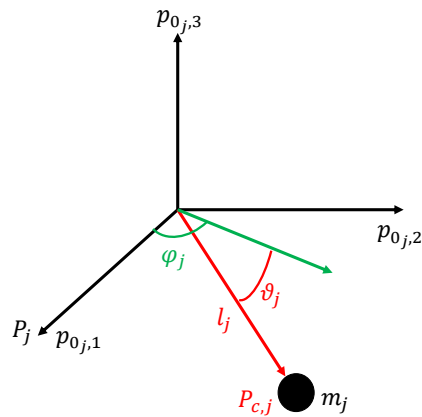


Fig. 2: Further detail of fuel slosh and reference frames

Using the variables and frames defined, the following section outlines the derivation of equations of motion for the spacecraft.

1.2 Derivation of Equations of Motion

1.2.1 Rigid Spacecraft Hub Translational Motion

The derivation begins with Newton's first law for the center of mass of the spacecraft.

$$\ddot{\mathbf{r}}_{C/N} = \frac{\mathbf{F}}{m_{sc}} \quad (1)$$

Ultimately the acceleration of the body frame or point B is desired

$$\ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \ddot{\mathbf{c}} \quad (2)$$

The definition of \mathbf{c} can be seen in Eq. (3).

$$\mathbf{c} = \frac{1}{m_{sc}} \left(m_{hub} \mathbf{r}_{Bc/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \right) \quad (3)$$

To find the inertial time derivative of \mathbf{c} , it is first necessary to find the time derivative of \mathbf{c} with respect to the body frame. A time derivative of any vector, \mathbf{v} , with respect to the body frame is denoted by \mathbf{v}' ; the inertial time derivative is labeled as $\dot{\mathbf{v}}$. The first and second body-relative time derivatives of \mathbf{c} can be seen in Eqs. (4) and (5).

$$\mathbf{c}' = \frac{1}{m_{sc}} \left(\sum_{j=1}^{N_P} m_j \mathbf{r}'_{P_{c,j}/B} \right) \quad (4)$$

$$\mathbf{c}'' = \frac{1}{m_{sc}} \left(\sum_{j=1}^{N_P} m_j \mathbf{r}''_{P_{c,j}/B} \right) \quad (5)$$

Remembering that the derivative of d is null respect to the body frame, the first and second body time derivatives of $\mathbf{r}_{P_{c,j}/B}$ are

$$\mathbf{r}_{P_{c,j}/B} = l_j \begin{bmatrix} \mathcal{P}_{0,j} \cos(\varphi_j) \cos(\vartheta_j) \\ \sin(\varphi_j) \cos(\vartheta_j) \\ -\sin(\vartheta_j) \end{bmatrix} + d \quad (6)$$

$$\mathbf{r}'_{P_{c,j}/B} = l_j \begin{bmatrix} \mathcal{P}_{0,j} [-\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j)] \\ \dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ -\dot{\vartheta}_j \cos(\vartheta_j) \end{bmatrix} \quad (7)$$

$$\mathbf{r}''_{P_{c,j}/B} = l_j \begin{bmatrix} \mathcal{P}_{0,j} [-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ \ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ -\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \end{bmatrix} \quad (8)$$

Eqs. (4) and (5) are next reformulated to include these new definitions:

$$\mathbf{c}' = \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(\dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} - \dot{\vartheta}_j \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right] \quad (9)$$

$$\mathbf{c}'' = \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \left(-\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right] \quad (10)$$

Using the transport theorem yields the following definition for $\ddot{\mathbf{c}}$

$$\ddot{\mathbf{c}} = \mathbf{c}'' + 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (11)$$

Eq. (2) is updated to include Eq. (11)

$$\ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \mathbf{c}'' - 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' - \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} - \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (12)$$

Substituting Eq.(10) into Eq.(12) results in

$$\ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \left(-\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right] - 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' - \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} - \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (13)$$

Moving second order terms to the left hand side and introducing the tilde matrix to replace the cross product operators simplifies the equation to

$$\begin{aligned} \ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}] \dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) + \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(-\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) + \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \ddot{\vartheta}_j \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' \\ - [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(-\dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \dot{\vartheta}_j^2 \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right] \end{aligned} \quad (14)$$

Rearranging the terms:

$$\begin{aligned}
\ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}]\dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \right. \\
+ \left. \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \right] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c}' \\
- [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \\
+ \left. \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
\left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \quad (15)
\end{aligned}$$

Equation (15) is the translational motion equation and is the first EOM needed to describe the motion of the spacecraft. The following section develops the rotational EOM.

1.2.2 Rigid Spacecraft Hub Rotational Motion

Starting with Euler's equation when the body fixed coordinate frame origin is not coincident with the center of mass of the body

$$\dot{\mathbf{H}}_{sc,B} = \mathbf{L}_B + m_{sc} \ddot{\mathbf{r}}_{B/N} \times \mathbf{c} \quad (16)$$

where \mathbf{L}_B is the total external torque about point B . The definition of the angular momentum vector of the spacecraft about point B is

$$\mathbf{H}_{sc,B} = [I_{hub,Bc}]\boldsymbol{\omega}_{B/N} + m_{hub} \mathbf{r}_{Bc/B} \times \dot{\mathbf{r}}_{Bc/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} \quad (17)$$

Now the inertial time derivative of Eq. (17) is taken and yields

$$\dot{\mathbf{H}}_{sc,B} = [I_{hub,Bc}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{hub,Bc}]\boldsymbol{\omega}_{B/N} + m_{hub} \mathbf{r}_{Bc/B} \times \ddot{\mathbf{r}}_{Bc/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \ddot{\mathbf{r}}_{P_{c,j}/B} \quad (18)$$

The terms $\ddot{\mathbf{r}}_{Bc/B}$ and $\ddot{\mathbf{r}}_{P_{c,j}/B}$ are found using the transport theorem and knowing that $\mathbf{r}_{Bc/B}$ is fixed with respect to the body frame.

$$\ddot{\mathbf{r}}_{Bc/B} = \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{Bc/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{Bc/B}) \quad (19)$$

$$\ddot{\mathbf{r}}_{P_{c,j}/B} = \mathbf{r}''_{P_{c,j}/B} + 2\boldsymbol{\omega}_{B/N} \times \mathbf{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}) \quad (20)$$

Incorporating Eqs. (19) - (20) into Eq. (18) results in

$$\begin{aligned}
\dot{\mathbf{H}}_{sc,B} = [I_{hub,Bc}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{hub,Bc}]\boldsymbol{\omega}_{B/N} + m_{hub} \mathbf{r}_{Bc/B} \times (\dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{Bc/B}) + \\
+ m_{hub} \mathbf{r}_{Bc/B} \times \left[\boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{Bc/B}) \right] + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \left[\mathbf{r}''_{P_{c,j}/B} + 2\boldsymbol{\omega}_{B/N} \times \mathbf{r}'_{P_{c,j}/B} + \right. \\
\left. + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}) \right] \quad (21)
\end{aligned}$$

Applying the parallel axis theorem the following inertia tensor terms are defined as

$$[I_{\text{hub},B}] = [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T \quad (22)$$

$$[I_{\text{sc},B}] = [I_{\text{hub},B}] + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \quad (23)$$

Taking the body-relative time derivative of Equation (23) yields

$$[I'_{\text{sc},B}] = - \sum_{j=1}^{N_P} m_j \left([\tilde{\mathbf{r}}'_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}] + [\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}'_{P_{c,j}/B}] \right) \quad (24)$$

The Jacobi Identity, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$, is applied to the cross products of Eq. (21):

$$\begin{aligned} \dot{\mathbf{H}}_{\text{sc},B} = & [I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{\text{hub},B_c}]\boldsymbol{\omega}_{B/N} - m_{\text{hub}}\mathbf{r}_{B_c/B} \times (\mathbf{r}_{B_c/B} \times \dot{\boldsymbol{\omega}}_{B/N}) + \\ & + m_{\text{hub}}\boldsymbol{\omega}_{B/N} \times \left[\mathbf{r}_{B_c/B} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{B_c/B}) \right] + \sum_{j=1}^{N_P} m_j \left\{ \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}''_{P_{c,j}/B} - \mathbf{r}_{P_{c,j}/B} \times (\mathbf{r}'_{P_{c,j}/B} \times \boldsymbol{\omega}_{B/N}) \right. \\ & - \mathbf{r}'_{P_{c,j}/B} \times (\mathbf{r}_{P_{c,j}/B} \times \boldsymbol{\omega}_{B/N}) + \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}'_{P_{c,j}/B}) \\ & \left. - \mathbf{r}_{P_{c,j}/B} \times (\mathbf{r}_{P_{c,j}/B} \times \dot{\boldsymbol{\omega}}_{B/N}) + \boldsymbol{\omega}_{B/N} \times \left[\mathbf{r}_{P_{c,j}/B} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}) \right] \right\} \quad (25) \end{aligned}$$

Rearranging the terms in Eq. (25) yields:

$$\begin{aligned} \dot{\mathbf{H}}_{\text{sc},B} = & \left\{ [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \right\} \dot{\boldsymbol{\omega}}_{B/N} \\ & + \boldsymbol{\omega}_{B/N} \times \left\{ [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \right\} \boldsymbol{\omega}_{B/N} \\ & + \sum_{j=1}^{N_P} m_j \left\{ \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}''_{P_{c,j}/B} - \left[[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}'_{P_{c,j}/B}] + [\tilde{\mathbf{r}}'_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \right] \boldsymbol{\omega}_{B/N} \right. \\ & \left. + \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}'_{P_{c,j}/B}) \right\} \quad (26) \end{aligned}$$

Using Eqs. (23) and (24) to simplify results in Eq. (26), the following simplified equation is obtained:

$$\begin{aligned} \dot{\mathbf{H}}_{\text{sc},B} = & [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{\text{sc},B}]\boldsymbol{\omega}_{B/N} + [I'_{\text{sc},B}]\boldsymbol{\omega}_{B/N} + \sum_{j=1}^{N_P} \left[m_j \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}''_{P_{c,j}/B} \right. \\ & \left. + m_j \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}'_{P_{c,j}/B}) \right] \quad (27) \end{aligned}$$

Eqs. (16) and (27) are equated and yield

$$\begin{aligned} \mathbf{L}_B + m_{\text{sc}}\ddot{\mathbf{r}}_{B/N} \times \mathbf{c} = & [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{\text{sc},B}]\boldsymbol{\omega}_{B/N} + [I'_{\text{sc},B}]\boldsymbol{\omega}_{B/N} + \sum_{j=1}^{N_P} \left[m_j \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}''_{P_{c,j}/B} \right. \\ & \left. + m_j \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}'_{P_{c,j}/B}) \right] \quad (28) \end{aligned}$$

Finally, using tilde matrix and simplifying yields the modified Euler equation, which is the second EOM necessary to describe the motion of the spacecraft.

$$[I_{sc,B}]\dot{\boldsymbol{\omega}}_{B/N} = -[\tilde{\boldsymbol{\omega}}_{B/N}][I_{sc,B}]\boldsymbol{\omega}_{B/N} - [I'_{sc,B}]\boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} \left(m_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}''_{P_{c,j}/B} + m_j [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} \right) + \mathbf{L}_B - m_{sc}[\tilde{\mathbf{c}}]\ddot{\mathbf{r}}_{B/N} \quad (29)$$

Rearranging Eq. (29) to be in the same form as the previous sections results in

$$\begin{aligned} & m_{sc}[\tilde{\mathbf{c}}]\ddot{\mathbf{r}}_{B/N} + [I_{sc,B}]\dot{\boldsymbol{\omega}}_{B/N} - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \right. \\ & + \left. \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \right] = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{sc,B}]\boldsymbol{\omega}_{B/N} - [I'_{sc,B}]\boldsymbol{\omega}_{B/N} \\ & - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \right. \\ & + \left. \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \right. \\ & \left. \left. \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \right\} \quad (30) \end{aligned}$$

1.2.3 Fuel Slosh Motion

The fuel slosh motion is being approximated by a lumped mechanical multi-mode model. Figure 1 shows that a single fuel slosh particle is free to move around the geometrical center of the tank at a fixed distance l_j and this formulation is generalized to include N_P number of fuel slosh particles. The derivation begins with Euler's law for each fuel slosh particle:

$$\dot{\mathbf{H}}_{T,j} = \mathbf{L}_{T,j} + m_j \dot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (31)$$

Where the $\mathbf{L}_{T,j}$ represents the external torques. It contains the damping term that is modeled as \mathbf{D} damping matrix multiplied by the relative velocity between the tank and the fuel, \mathbf{l}'_j . It takes into account the torque due to gravity and any other external torque.

$$\mathbf{L}_{T,j} = \mathbf{l}_j \times \mathbf{F}_g - \mathbf{D} \mathbf{l}'_j + \boldsymbol{\tau}_{ext,j} \quad (32)$$

It is necessary to express \mathbf{H}_T of the fuel slosh particle as:

$$\mathbf{H}_{T,j} = m_j \mathbf{l}_j \times \dot{\mathbf{l}}_j \quad (33)$$

Deriving this equation we obtain:

$$\dot{\mathbf{H}}_{T,j} = m_j \mathbf{l}_j \times \ddot{\mathbf{l}}_j \quad (34)$$

Using the transport theorem once again:

$$\ddot{\mathbf{l}}_j = \mathbf{l}''_j + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \quad (35)$$

\mathbf{l}_j prime and second derivative are equal to (7) and (8) because the \mathbf{d} vector is constant in body frame. The previous results are reported here:

$$\mathbf{l}_j = l_j \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{array}{c} \cos(\varphi_j) \cos(\vartheta_j) \\ \sin(\varphi_j) \cos(\vartheta_j) \\ -\sin(\vartheta_j) \end{array} \right] \end{matrix} \quad (36)$$

$$\mathbf{l}'_j = l_j \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{array}{c} -\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ \dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ -\dot{\vartheta}_j \cos(\vartheta_j) \end{array} \right] \end{matrix} \quad (37)$$

$$\mathbf{l}''_j = l_j \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{array}{c} -\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ \ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ -\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \end{array} \right] \end{matrix} \quad (38)$$

Equating eqs. (31) and (34) and using eq. (35), we obtain:

$$m_j \mathbf{l}_j \times [\mathbf{l}''_j + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j)] = \mathbf{L}_{T,j} + m_j \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (39)$$

Substituting eq. (38) in eq. (39)

$$\begin{aligned} m_j \mathbf{l}_j \times \left[l_j \left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \right. \right. \\ \left. \left. + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + l_j \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \right. \right. \\ \left. \left. - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + l_j \left(-\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right. \\ \left. + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] = \mathbf{L}_{T,j} + m_j \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (40) \end{aligned}$$

Execute separately the following vectorial product, using C and S to indicate cos and sin, respectively.

$$\begin{aligned}
& \mathcal{P}_{0,j} \begin{bmatrix} 0 & S(\vartheta_j) & S(\varphi_j)C(\vartheta_j) \\ -S(\vartheta_j) & 0 & -C(\varphi_j)C(\vartheta_j) \\ -S(\varphi_j)C(\vartheta_j) & C(\varphi_j)C(\vartheta_j) & 0 \end{bmatrix} \mathcal{P}_{0,j} \begin{bmatrix} -\ddot{\varphi}_j S(\varphi_j)C(\vartheta_j) - \ddot{\vartheta}_j C(\varphi_j)S(\vartheta_j) \\ -\dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j) - \vartheta_j^2 C(\varphi_j)C(\vartheta_j) \\ +2\dot{\varphi}_j\dot{\vartheta}_j S(\varphi_j)S(\vartheta_j) \\ \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)S(\vartheta_j) \\ -\dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j) - \vartheta_j^2 S(\varphi_j)C(\vartheta_j) \\ -2\dot{\varphi}_j\dot{\vartheta}_j C(\varphi_j)S(\vartheta_j) \\ -\ddot{\vartheta}_j C(\vartheta_j) + \dot{\vartheta}_j^2 S(\vartheta_j) \end{bmatrix} \\
& = l_j^2 \mathcal{P}_{0,j} \begin{bmatrix} \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) - \dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \vartheta_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ -2\dot{\varphi}_j\dot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)C^2(\vartheta_j) + \vartheta_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ \ddot{\varphi}_j S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) + \dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \vartheta_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ -2\dot{\varphi}_j\dot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)C^2(\vartheta_j) - \vartheta_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ \ddot{\varphi}_j S^2(\varphi_j)C^2(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \dot{\varphi}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) \\ + \vartheta_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) - 2\dot{\varphi}_j\dot{\vartheta}_j S^2(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\varphi}_j C^2(\varphi_j)C^2(\vartheta_j) \\ - \ddot{\vartheta}_j C(\varphi_j)S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \dot{\varphi}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) - \vartheta_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) \\ - 2\dot{\varphi}_j\dot{\vartheta}_j C^2(\varphi_j)C(\vartheta_j)S(\vartheta_j) \end{bmatrix} \\
& = l_j^2 \mathcal{P}_{0,j} \begin{bmatrix} \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j) - \dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - 2\dot{\varphi}_j\dot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) \\ \ddot{\varphi}_j S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j) + \dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - 2\dot{\varphi}_j\dot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) \\ \ddot{\varphi}_j C^2(\vartheta_j) - 2\dot{\varphi}_j\dot{\vartheta}_j C(\vartheta_j)S(\vartheta_j) \end{bmatrix} \quad (41)
\end{aligned}$$

Using the results obtained in eq. (41), project eq. (40) on the axis of rotation of φ and ϑ . The φ rotation axis is $\hat{\mathbf{p}}_{0,j,3}$, while we will call $\hat{\mathbf{p}}'_{0,j,2}$ the ϑ rotation axis.

$$\hat{\mathbf{p}}_{0,j,3} = \mathcal{P}_{0,j} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (42)$$

$$\begin{aligned}
m_j l_j^2 \left[\ddot{\varphi}_j \cos^2(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\vartheta_j) \sin(\vartheta_j) \right] + m_j \hat{\mathbf{p}}_{0,j,3}^T \mathbf{l}_j \times \left[\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] \\
= \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + m_j \hat{\mathbf{p}}'_{0,j,2} \ddot{\vartheta}_j \mathbf{r}_{T/N} \times \mathbf{l}_j \quad (43)
\end{aligned}$$

$$\hat{\mathbf{p}}'_{0,j,2} = \mathcal{P}_{0,j} \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \quad (44)$$

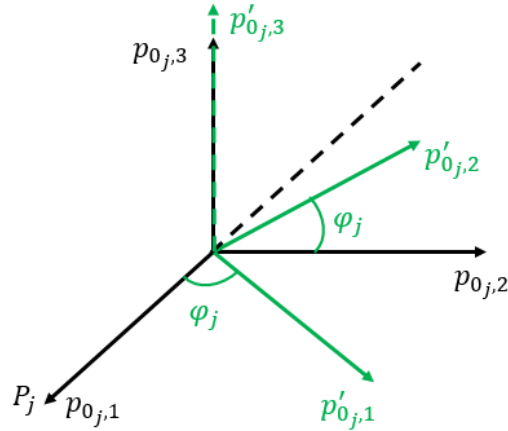


Fig. 3: Rotation Axes

$$m_j l_j^2 \left[\ddot{\vartheta}_j + \dot{\varphi}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) \right] + m_j \hat{\mathbf{p}}_{0,j,2}'^T \mathbf{l}_j \times \left[\dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] = \hat{\mathbf{p}}_{0,j,2}'^T \mathbf{L}_{T,j} + m_j \hat{\mathbf{p}}_{0,j,2}'^T \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (45)$$

Remembering that $\mathbf{r}_{T/N} = \mathbf{r}_{B/N} + \mathbf{d}$, and that \mathbf{d} is constant in the body frame, we can use once again the transport theorem to write:

$$\ddot{\mathbf{d}} = \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{d} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{d}) \quad (46)$$

$$\ddot{\mathbf{r}}_{T/N} = \ddot{\mathbf{r}}_{B/N} + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{d} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{d}) \quad (47)$$

Rearranging the terms as done in the previous sections

$$m_j l_j^2 \ddot{\varphi}_j \cos^2(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} + m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} = -m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + 2m_j l_j^2 \dot{\varphi}_j \dot{\vartheta}_j \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \quad (48)$$

$$m_j l_j^2 \ddot{\vartheta}_j - m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} + m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} = -m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,2}'^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\varphi}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \quad (49)$$

Eqs. (48) and (49) are the Fuel Slesh Particle equations.

1.3 Back-substitution Method

The equations presented in the previous sections result in $2N_P + 6$ coupled differential equations. Therefore, if the EOMs were placed into state space form, a system mass matrix of size $2N_P + 6$ would need to be inverted to numerically integrate the EOMs. This can result in a computationally expensive simulation. The computation effort to numerically invert an $N \times N$ matrix scales with N^3 . In the following section, the EOMs are manipulated using a back-substitution method to increase the computational efficiency.

This manipulation involves inverting twice a (3×3) matrix, the A^{-1} and $(D - CA^{-1}B)^{-1}$ matrices as it is shown in eqs. (75) and (76). Then the system is completely solved back substituting for fuel slosh and translational motions. The derivation of the back-substitution method can be seen in the following sections.

1.3.1 Fuel Slosh Motion

Starting from eq. (48)

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left\{ m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + 2m_j l_j^2 \dot{\vartheta}_j \dot{\vartheta} \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (50)$$

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left(m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} + a_{\vartheta_j} \right) \quad (51)$$

Writing it this way instead

$$\ddot{\vartheta}_j = \mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j} \quad (52)$$

Where

$$\mathbf{a}_{\vartheta_j}^T = -\frac{\hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j]}{l_j^2 \cos^2(\vartheta_j)} \quad (53)$$

$$\mathbf{b}_{\vartheta_j}^T = \frac{\hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}])}{l_j^2 \cos^2(\vartheta_j)} \quad (54)$$

$$c_{\vartheta_j} = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left\{ -m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + 2m_j l_j^2 \dot{\vartheta}_j \dot{\vartheta} \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (55)$$

Doing the same for eq. (49)

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2} \left\{ m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,2}^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\vartheta}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (56)$$

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2} \left(m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} + a_{\vartheta_j} \right) \quad (57)$$

Writing this a different way

$$\ddot{\vartheta}_j = \mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j} \quad (58)$$

Where

$$\mathbf{a}_{\vartheta_j}^T = -\frac{\hat{\mathbf{p}}_{0,j,2}'^T[\tilde{\mathbf{l}}_j]}{l_j^2} \quad (59)$$

$$\mathbf{b}_{\vartheta_j}^T = \frac{\hat{\mathbf{p}}_{0,j,2}'^T[\tilde{\mathbf{l}}_j](\tilde{\mathbf{l}}_j + [\tilde{\mathbf{d}}])}{l_j^2} \quad (60)$$

$$c_{\vartheta_j} = \frac{1}{m_j l_j^2} \left\{ -m_j \hat{\mathbf{p}}_{0,j,2}'^T[\tilde{\mathbf{l}}_j][\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{d} + \hat{\mathbf{p}}_{0,j,2}'^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\varphi}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) \right. \\ \left. - m_j \hat{\mathbf{p}}_{0,j,2}'^T[\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{l}_j \right] \right\} \quad (61)$$

1.3.2 Translation

Plugging these definitions into the translation equation

$$\ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}]\dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \right. \\ \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \right] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c}' \\ - [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \\ \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\ \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \quad (62)$$

Results in

$$\ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}]\dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) (\mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j}) \right. \\ \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) (\mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j}) \right] \\ = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c}' - [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}]\mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \\ \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\ \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \quad (63)$$

Simplifying

$$\begin{aligned}
& \left\{ [I_{3 \times 3}] - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. \left. + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \ddot{\mathbf{r}}_{B/N} + \left\{ -[\tilde{\mathbf{c}}] - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
& \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (64)
\end{aligned}$$

Multiply both sides by m_{sc} .

$$\begin{aligned}
& \left\{ m_{\text{sc}} [I_{3 \times 3}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. \left. + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \ddot{\mathbf{r}}_{B/N} + \left\{ -m_{\text{sc}} [\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = m_{\text{sc}} \ddot{\mathbf{r}}_{C/N} - 2m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
& \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (65)
\end{aligned}$$

1.3.3 Rotation

Same thing for rotation:

$$\begin{aligned}
& m_{sc}[\tilde{\mathbf{c}}]\ddot{\mathbf{r}}_{B/N} + [I_{sc,B}]\dot{\boldsymbol{\omega}}_{B/N} - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) (\mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j}) + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right. \right. \\
& \left. \left. + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) (\mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j}) \right] = \\
& \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{sc,B}]\boldsymbol{\omega}_{B/N} - [I'_{sc,B}]\boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}]\mathbf{r}'_{P_{c,j}/B} \right. \\
& \left. + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \right\} \\
& \hspace{15em} (66)
\end{aligned}$$

Next

$$\begin{aligned}
& \left\{ m_{sc}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \ddot{\mathbf{r}}_{B/N} + \left\{ [I_{sc,B}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{sc,B}]\boldsymbol{\omega}_{B/N} - [I'_{sc,B}]\boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}]\mathbf{r}'_{P_{c,j}/B} \right. \\
& \left. + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \left. \left. - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\
& \left. \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \right\} \hspace{1em} (67)
\end{aligned}$$

1.3.4 Remaining Back-substitution Steps

The following definitions can be defined:

$$\begin{aligned}
[A] = & \left\{ m_{sc}[I_{3 \times 3}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T \right. \right. \\
& \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \hspace{1em} (68)
\end{aligned}$$

$$[B] = \left\{ -m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T \right. \right. \\ \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \quad (69)$$

$$[C] = \left\{ m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T \right. \right. \\ \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \quad (70)$$

$$[D] = \left\{ [I_{\text{sc},B}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T \right. \right. \\ \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \quad (71)$$

$$\mathbf{v}_{\text{trans}} = m_{\text{sc}} \ddot{\mathbf{r}}_{C/N} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} \\ - \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\ \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\ \left. \left. - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\ \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (72)$$

$$\mathbf{v}_{\text{rot}} = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} \\ - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \right. \\ \left. \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\ \left. \left. - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\ \left. \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \right\} \quad (73)$$

Therefore the translation and rotation EOMs are written in the following form

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{B/N} \\ \tilde{\boldsymbol{\omega}}_{B/N} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}} \end{bmatrix} \quad (74)$$

Solving the system-of-equations by

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \left([D] - [C][A]^{-1}[B] \right)^{-1} (\mathbf{v}_{\text{rot}} - [C][A]^{-1}\mathbf{v}_{\text{trans}}) \quad (75)$$

$$\ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} = [A]^{-1}(\mathbf{v}_{\text{trans}} - [B]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}) \quad (76)$$

Now the other state variables can be solved using Eqs. (52) and (58):

$$\ddot{\varphi}_j = \mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\varphi_j} \quad (77)$$

$$\ddot{\vartheta}_j = \mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\vartheta_j} \quad (78)$$

1.4 Rotational Kinetic Energy

The total rotational kinetic energy (i.e. kinetic energy about the center of mass) of the spacecraft is:

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, C} \cdot \dot{\mathbf{r}}_{B_c, C} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/C} \cdot \dot{\mathbf{r}}_{P_{c,j}/C} \quad (79)$$

Expanding these terms results in

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} (\dot{\mathbf{r}}_{B_c, B} - \dot{\mathbf{c}}) \cdot (\dot{\mathbf{r}}_{B_c, B} - \dot{\mathbf{c}}) + \sum_{j=1}^{N_P} \frac{1}{2} m_j (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \cdot (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \quad (80)$$

Expanding further

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} (\dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} - 2\dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{c}} + \dot{\mathbf{c}} \cdot \dot{\mathbf{c}}) \\ + \sum_{j=1}^{N_P} \frac{1}{2} m_j (\dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} - 2\dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{c}} + \dot{\mathbf{c}} \cdot \dot{\mathbf{c}}) \quad (81)$$

Combining like terms results in

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} \\ - \left[m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \right] \cdot \dot{\mathbf{c}} + \frac{1}{2} \left[m_{\text{hub}} + \sum_{j=1}^{N_P} m_j \right] \dot{\mathbf{c}} \cdot \dot{\mathbf{c}} \quad (82)$$

Performing a final simplification yields

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} - \frac{1}{2} m_{SC} \dot{\mathbf{c}} \cdot \dot{\mathbf{c}} \quad (83)$$

1.5 Rotational Angular Momentum

The total rotational angular momentum of the spacecraft about point C is

$$\mathbf{H}_{rot,C} = [I_{hub,B_c}] \boldsymbol{\omega}_{B/N} + m_{hub} \mathbf{r}_{B_c,C} \times \dot{\mathbf{r}}_{B_c,C} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/C} \times \dot{\mathbf{r}}_{P_{c,j}/C} \quad (84)$$

Expanding these terms yields

$$\mathbf{H}_{rot,C} = [I_{hub,B_c}] \boldsymbol{\omega}_{B/N} + m_{hub} (\mathbf{r}_{B_c,B} - \mathbf{c}) \times (\dot{\mathbf{r}}_{B_c,B} - \dot{\mathbf{c}}) + \sum_{j=1}^{N_P} m_j (\mathbf{r}_{P_{c,j}/B} - \mathbf{c}) \times (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \quad (85)$$

Distributing this result

$$\begin{aligned} \mathbf{H}_{rot,C} = & [I_{hub,B_c}] \boldsymbol{\omega}_{B/N} + m_{hub} (\mathbf{r}_{B_c,B} \times \dot{\mathbf{r}}_{B_c,B} - \mathbf{r}_{B_c,B} \times \dot{\mathbf{c}} - \mathbf{c} \times \dot{\mathbf{r}}_{B_c,B} + \mathbf{c} \times \dot{\mathbf{c}}) \\ & + \sum_{j=1}^{N_P} m_j (\mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} - \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{c}} - \mathbf{c} \times \dot{\mathbf{r}}_{P_{c,j}/B} + \mathbf{c} \times \dot{\mathbf{c}}) \end{aligned} \quad (86)$$

Simplifying this result yields the final equation

$$\mathbf{H}_{rot,C} = [I_{hub,B_c}] \boldsymbol{\omega}_{B/N} + m_{hub} \mathbf{r}_{B_c,B} \times \dot{\mathbf{r}}_{B_c,B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} - m_{SC} \mathbf{c} \times \dot{\mathbf{c}} \quad (87)$$

1.6 Reference System Change

As we can see in Eqs. (53), (54) and (55), the problem is singular for $\vartheta = \pi/2 + k\pi$, ($k = 0, 1, 2, \dots$). In order to solve this issue, a rotation of the pendulum frame, $\mathcal{P}_{0,j}$, is necessary. This rotation is performed when ϑ_j reaches a value multiple of $\pi/4$, to remain always far enough from the singularity. The new pendulum frame $\mathcal{P}_{0,j}^{new}$ is obtained with a rotation of the actual value of φ around the $\hat{\mathbf{p}}_{3,j}$ axes, and of ϑ around $\hat{\mathbf{p}}_{2,j}$ axes. This would lead to a new pendulum reference frame with $\hat{\mathbf{p}}_{1,j}^{new}$ aligned along the pendulum direction. The Fig. 4 shows the $\mathcal{P}_{0,j}^{new}$ respect to $\mathcal{P}_{0,j}$. At this point is easy to see that the new value of φ_j and ϑ_j are equal to 0. To compute the new value of $\dot{\varphi}_j$ and $\dot{\vartheta}_j$, Eq. (37) is reversed and it yields:

$$\dot{\varphi} = \frac{\mathbf{l}_j[2]}{l_j} \quad (88)$$

$$\dot{\vartheta} = -\frac{\mathbf{l}_j[3]}{l_j} \quad (89)$$

The integration can continue using these new values and the new reference systems. This would lead to discontinuities on φ_j , θ_j , $\dot{\varphi}_j$ and $\dot{\vartheta}_j$ but not on the vectors \mathbf{l}_j and \mathbf{l}'_j .

2 Model Functions

This module is intended to be used as a oscillating spherical pendulum attached to the spacecraft. Below is a list of functions that this model performs:

- Compute it's contributions to the mass properties of the spacecraft
- Provides matrix contributions for the back substitution method
- Compute it's derivatives
- Adds energy and momentum contributions to the spacecraft

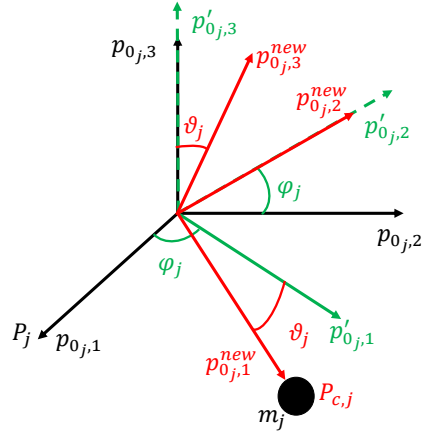


Fig. 4: $\mathcal{P}_{0,j}^{new}$ frame definition

3 Model Assumptions and Limitations

Below is a summary of the assumptions/limitations:

- The mass is assumed to be concentrated at the pendulum meaning that the mass is considered a point mass
- The link is considered massless: the link being the length between the rotation point and the mass
- Is derived in such a manner that does not require constraints to be met

4 Test Description and Success Criteria

This test is located in `simulation/dynamics/sphericalPendulum/UnitTest/test_sphericalPendulum.py`. In this integrated test there are two spherical pendulums connected to the spacecraft hub. There are two scenarios being tested: one with the time step being 0.01 and one with the time step 0.001. Both tests should show energy and momentum conservation but the 0.001 should show less integration error.

5 Test Parameters

To validate and test the code a simulation with two spherical pendulums has been run using the following input parameters:

$$\begin{aligned}
 l_1 &= 0.3m, & m_1 &= 20kg, & \dot{\varphi}_{1,0} &= 0.01rad/s, & \dot{\vartheta}_{1,0} &= 0.05rad/s \\
 \mathbf{d}_1 &= \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} m & \hat{\mathbf{p}}_{0,1} &= \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} & \hat{\mathbf{p}}_{0,2} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \hat{\mathbf{p}}_{0,3} &= \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}
 \end{aligned} \quad (90)$$

$$\begin{aligned}
 l_2 &= 0.4m, & m_2 &= 40kg, & \dot{\varphi}_{2,0} &= 0.1rad/s, & \dot{\vartheta}_{2,0} &= 0.5rad/s \\
 \mathbf{d}_1 &= \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} m & \hat{\mathbf{p}}_{0,1} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \hat{\mathbf{p}}_{0,2} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \hat{\mathbf{p}}_{0,3} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned} \quad (91)$$

Table 2: Error Tolerance - Note: Relative Tolerance is $\text{abs}(\frac{\text{truth}-\text{value}}{\text{truth}})$

Test	Relative Tolerance
Energy and Momentum Conservation	1e-8

6 Test Results

The following figures show the conservation of the quantities described in the success criteria for each scenario. The conservation plots are all relative difference plots. All conservation plots show integration error which is the desired result. In the python test these values are automatically checked therefore when the tests pass, these values have all been confirmed to be conserved. An additional note: the angular momentum plots are plotting the change in the components of the angular momentum vector in the inertial frame. The individual components are not labeled because the goal is for each component to show conservation therefore the individual components do not have separate information needing to be specified.

6.1 Time Step = 0.01

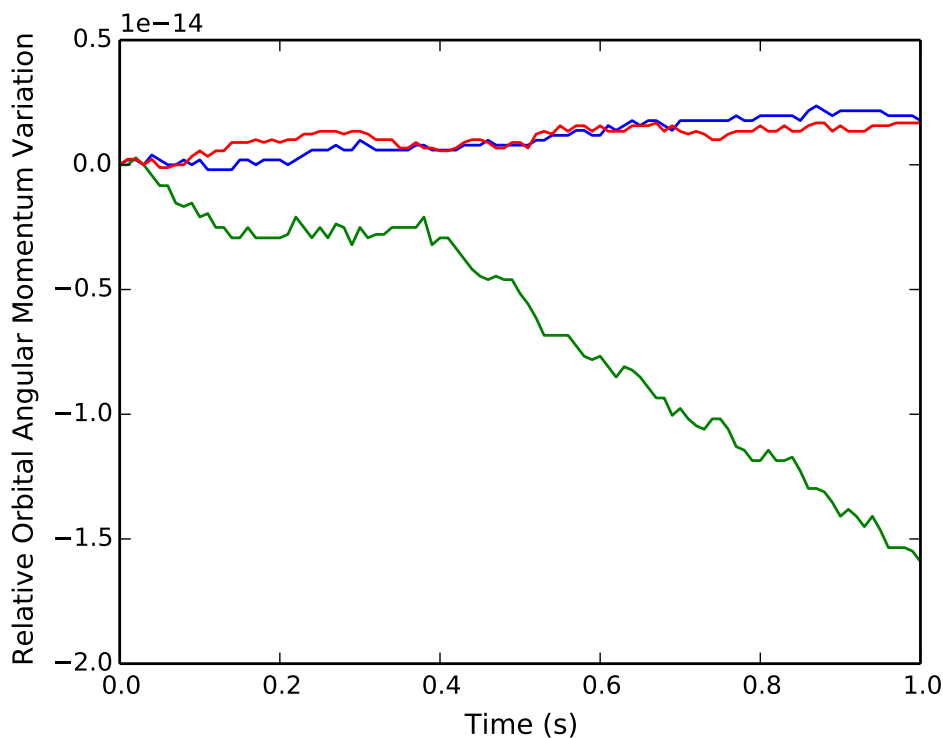


Fig. 5: Change in Orbital Angular Momentum Time Step = 0.01

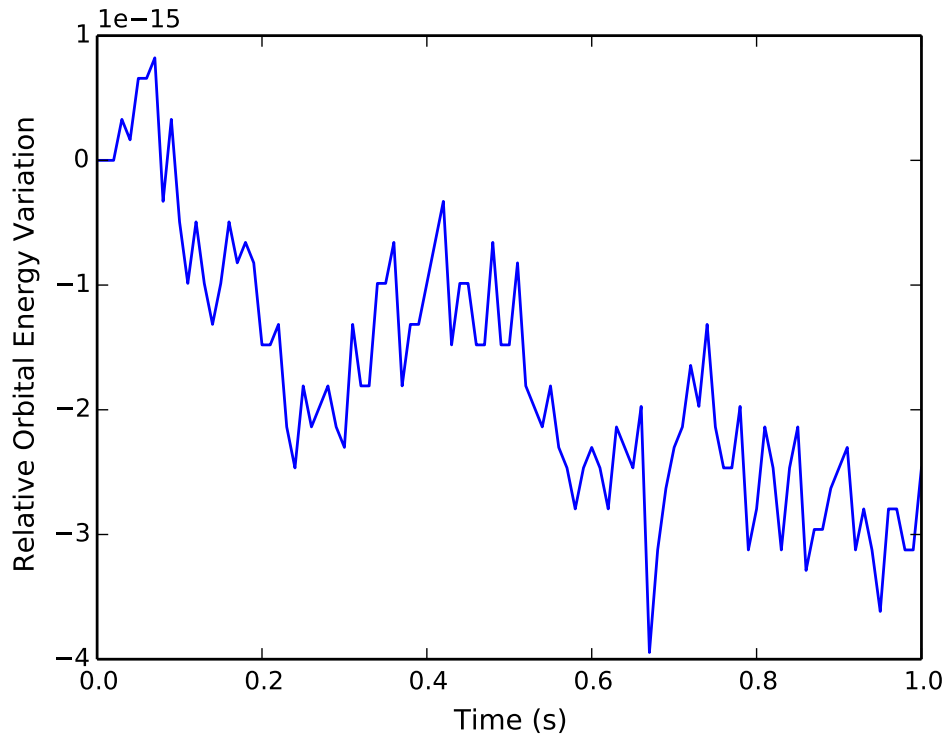


Fig. 6: Change in Orbital Energy Time Step = 0.01

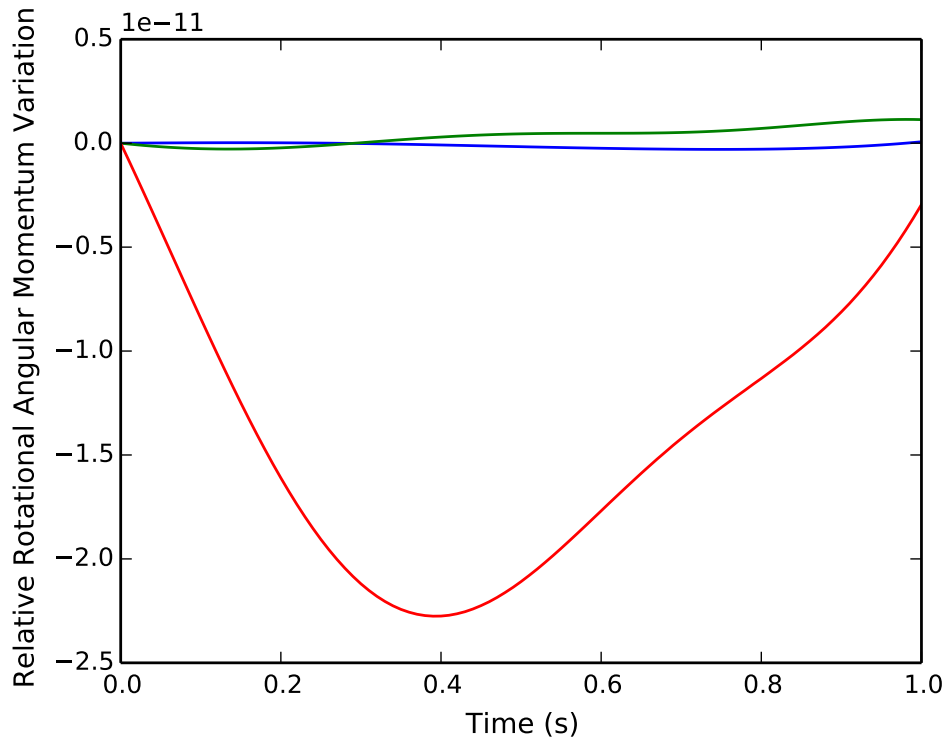


Fig. 7: Change In Rotational Angular Momentum Time Step = 0.01

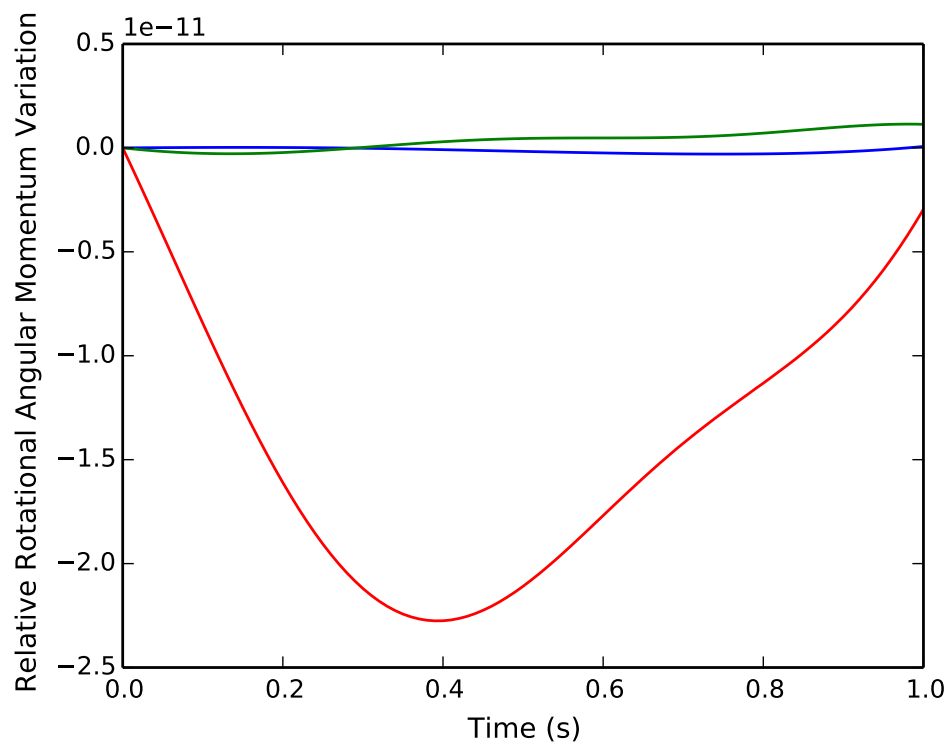


Fig. 8: Change In Rotational Energy Time Step = 0.01

6.2 Time Step = 0.001

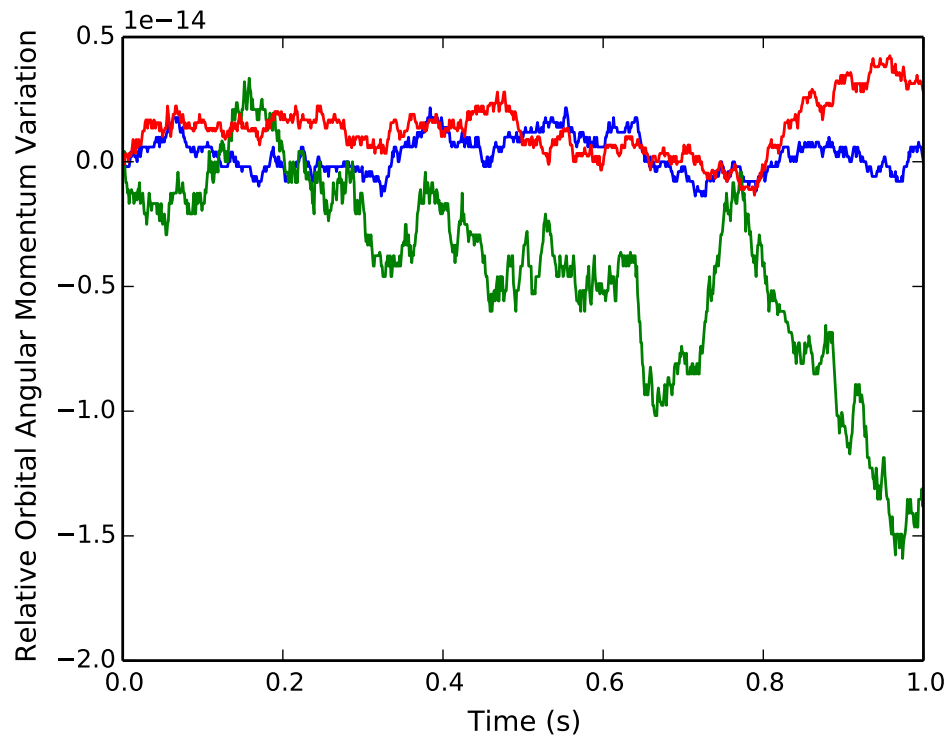


Fig. 9: Change in Orbital Angular Momentum Time Step = 0.001

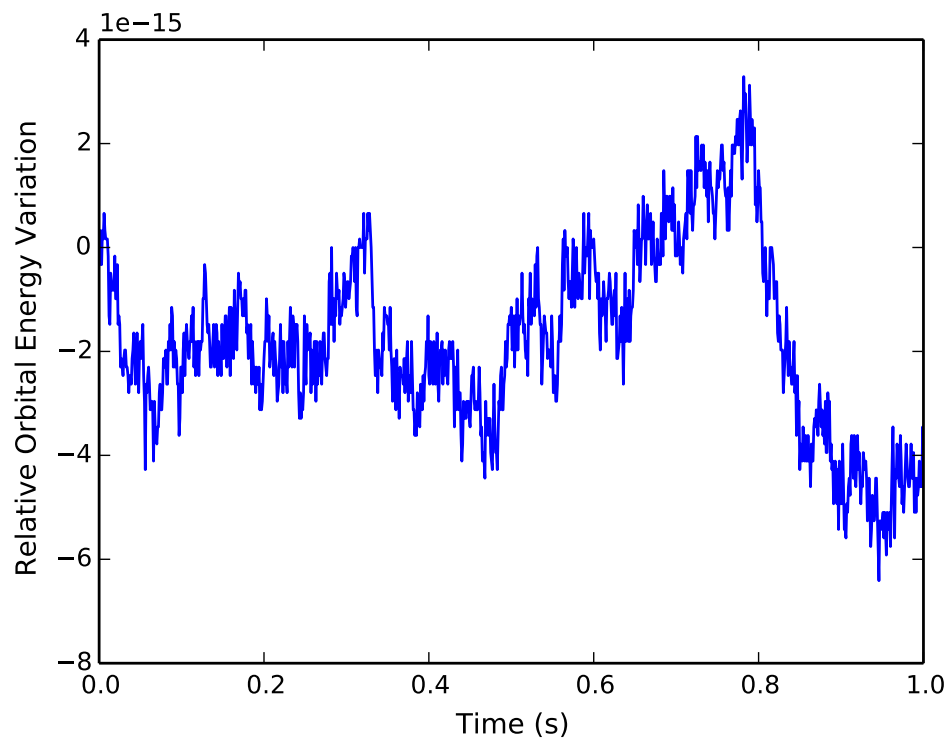


Fig. 10: Change in Orbital Energy Time Step = 0.001

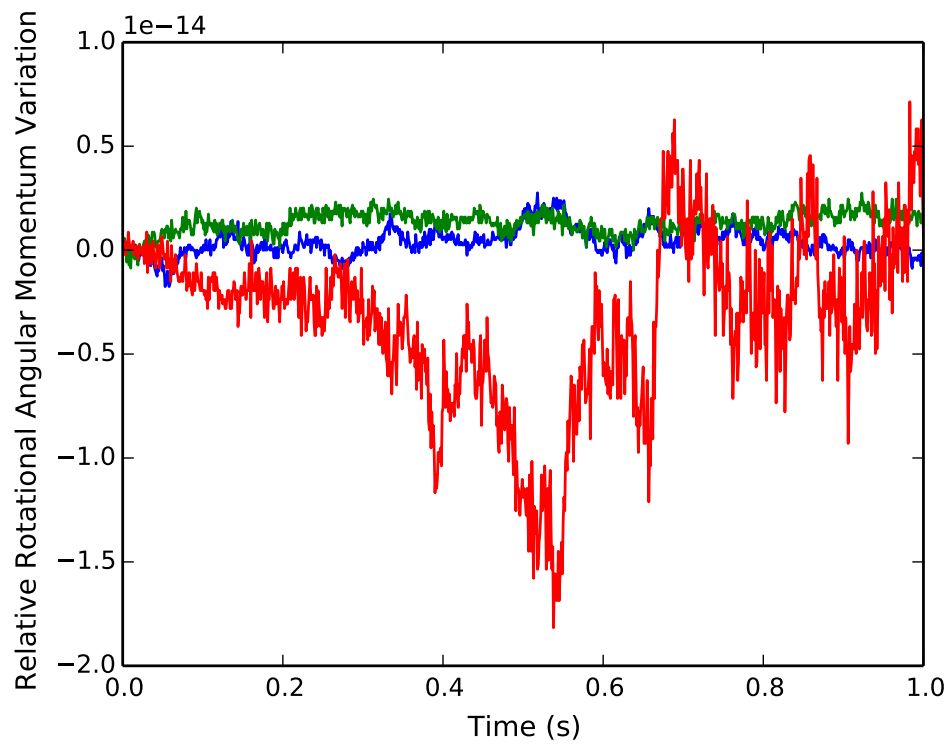


Fig. 11: Change In Rotational Angular Momentum Time Step = 0.001

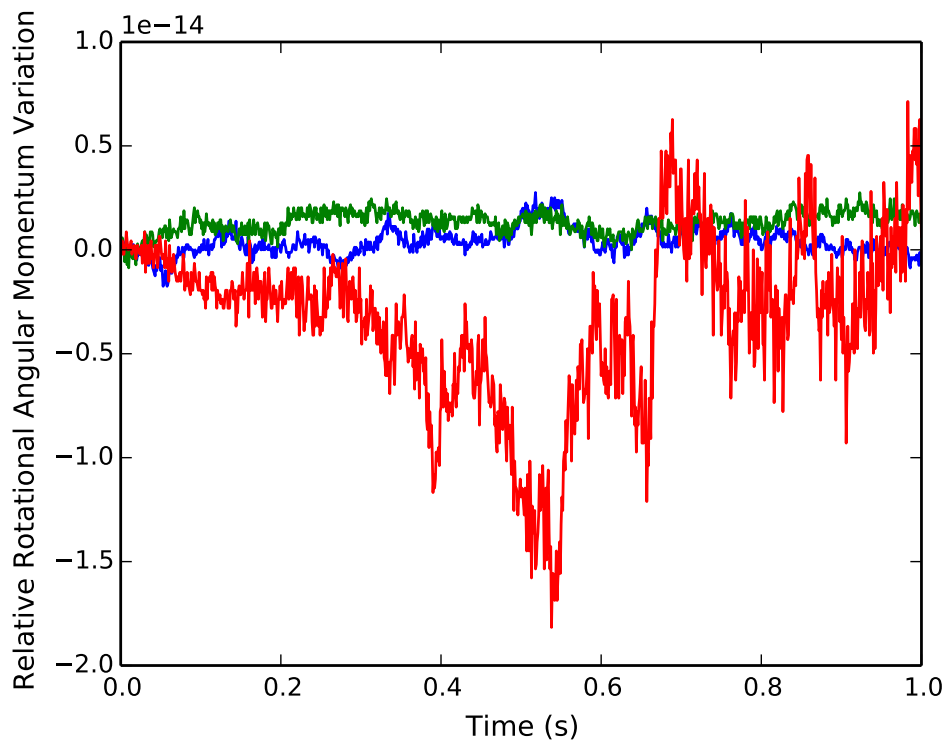


Fig. 12: Change In Rotational Energy Time Step = 0.001

7 User Guide

This section is to outline the steps needed to setup a spherical pendulum in python using Basilisk.

1. Import the sphericalPendulum class:

```
from Basilisk.simulation import sphericalPendulum
```

2. Create an instantiation of a spherical pendulum particle:

```
particle1 = sphericalPendulum.SphericalPendulum()
```

3. Define all physical parameters for a spherical pendulum particle. For example:

```
particle1.r_PB_B = [[0.1], [0], [-0.1]] Do this for all of the parameters for a spherical pendulum  
seen in the public variables in the .h file.
```

4. Define the initial conditions of the states:

```
particle1.phiInit = 0.05 particle1.phiDotInit = 0.0
```

5. Define a unique name for each state:

```
particle1.nameOfPhiState = "sphericalPendulumPhi" particle1.nameOfPhiDotState = "spher-  
icalPendulumPhiDot"
```

6. Finally, add the particle to your spacecraftPlus:

```
scObject.addStateEffector(particle1). See spacecraftPlus documentation on how to set up a space-  
craftPlus object.
```

REFERENCES