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EFFECTOR DYNAMICS FOR SEQUENTIALLY ROTATING RIGID BODY SPACECRAFT COMPONENTS

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Spacecraft simulations and subsequent analysis are critical steps in any mission. The first step is typically to develop the equations of motion that describe the behavior of the space vehicle such that they can be simulated to verify mission requirements. However, with increasingly more complex spacecraft configurations, developing and validating such equations of motion is difficult and often spacecraft-specific. Robotic arm appendages are particularly interesting, as they have been used to service and dock spacecraft, which is the primary mission of many proposed missions. Prior work has focused on developing the Back-Substitution Method (BSM) to modularly create the equations of motion under the assumption that the spacecraft is composed of a rigid hub to which multiple rigid bodies (effectors) are attached. These can include solar panels, reaction wheels, etc. The BSM formulation has been expanded to simulate general single and dual-axis rotating rigid bodies with any mass distribution and spin axis. It creates a singular formulation that can represent a plethora of effectors, from control moment gyroscopes to dual-gimbal antennas under the same equations of motion. This paper aims to expand this work further to simulate a chain of rotating rigid bodies of any length, not constrained to one or two axes. Kane's equations are used to identify patterns in the equations of motion development that allow for this elegant generalization. The rigid bodies can be connected by any sequence of single-axis rotations, vastly improving the possible spacecraft configurations and allowing for the simulation of very complex robotic arms attached to a rigid hub.

INTRODUCTION

Space vehicles have become increasingly complex to keep up with ambitious mission requirements. Starting with Sputnik 1,[1] the first artificial satellite composed of little more than a sphere with some sensors and four boom antennas, spacecraft design has changed dramatically as technology has evolved. The International Space Station[2] (ISS) has been in orbit for more than two decades and is arguably the most complex space vehicle of the twenty-first century. Notably, many spacecraft now include multi-link robotic arms to perform in-orbit operations like servicing and docking. The Canadarm[3–5] has been used by the Space Shuttle (Canadarm 1) and the ISS (Canadarm 2), while the Lunar Gateway will also make use of it (Canadarm 3). Other servicing vehicles like Northrop Grumman's Mission Extension Vehicle[6] or the National Air and Space Administration's (NASA) OSAM-1 Mission[7] also make use of robotic arms for life-extension and servicing missions. The need for robotic manipulators in space is clear, and the future of the industry will rely on them for future missions.

The vehicle must be simulated and analyzed to ensure the spacecraft meets mission requirements. These steps require the equations of motion that describe the vehicle's behavior in orbit to be derived and validated. However, simulating multi-link appendages is challenging due to their multiple degrees of freedom and the

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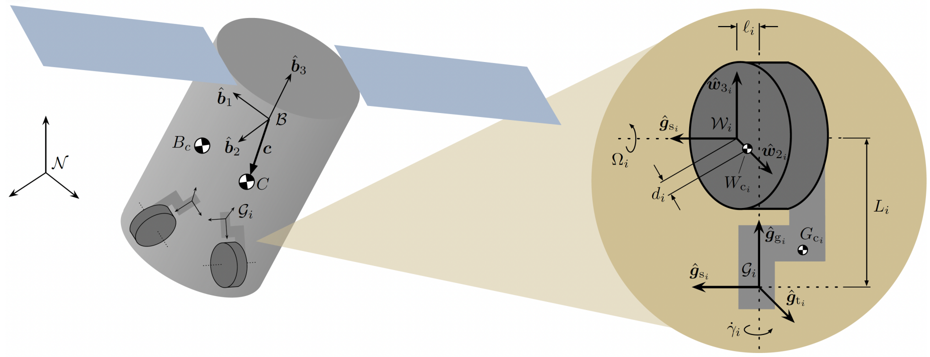
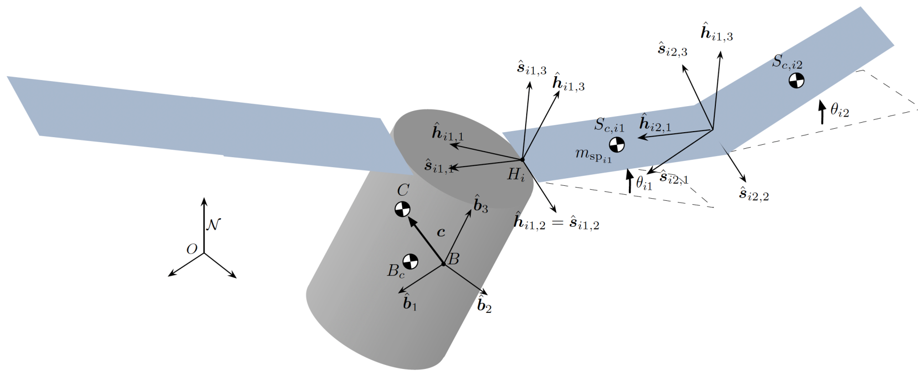


(a) Diagram for a reaction wheel effector.[9]



(b) Diagram for a solar panel effector.[10,11]

Figure 1: Diagrams for two different single-axis components.

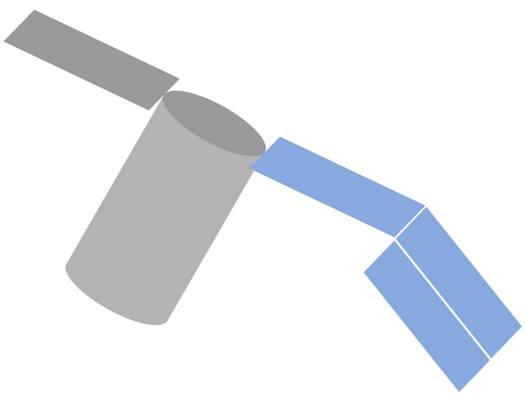
inherent dynamic coupling between links. One common approach in robotics is to assume that the robotic arm is prescribed;[8] that is, to suppose that each degree of freedom is precisely imposed by an actuator/motor. The output of all the actuators would describe the motion of the arm. Effectively, this means that the arm behaves separately from the rigid hub, not being affected by it; however, the rigid hub is still affected by the motion of the arm. This one-way coupling simplifies the equations of motion and is a fair assumption in many applications. Nevertheless, this is still an approximation and standard validation metrics like energy and angular momentum conservation cannot be used to verify that the simulation is correct. Moreover, even when the full coupling equations of motion are used, they are often developed for a particular spacecraft configuration or specific types of effectors. This lack of generality means that the equations of motion must be received and invalidated each time the configuration changes, slowing progress.

Previous work introduced the Back-Substitution Method (BSM), which rearranges the equations of motion of each effector and the hub to fit a standard form. With BSM, effectors are the dynamic components added to a central, rigid hub component. In turn, instead of having to invert a large system mass matrix, which is computationally very expensive, BSM first solves for the translational and rotational accelerations of the hub $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$, respectively, by having already analytically back-substituted the effector equations of motion into the hub differential equations. Next, the degrees of freedom of each effector are solved with the hub accelerations.[10,12] BSM has been used to model several different effectors, including balanced and imbalanced reaction wheels,[13] variable-speed control moment gyroscopes,[9] and hinged solar arrays.[11] The modeling of these effectors starts by making assumptions about each component, especially concerning the spin axis and the body's mass distribution. To illustrate this, Figure 1 shows diagrams of two rotating dual-axis components: Figure 1a shows the problem statement of a spacecraft with a variable-speed control moment gyroscope, and Figure 1b shows the problem statement of a spacecraft with a dual-hinged panel.

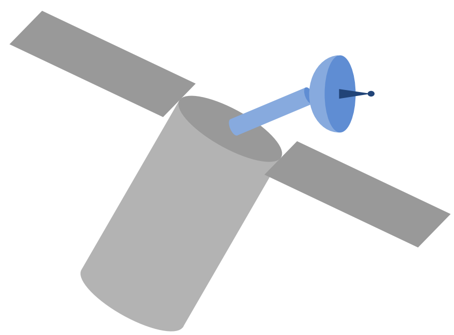
In these examples, both effectors represent dual-axis rotating rigid bodies. However, because of their initial assumptions, the resulting equations of motion are different, as some terms vanish because of the choice of mass distribution and spin axis. While this formulation still describes each particular effector, it lacks generality; one standard formulation could have been used with general spin axes and mass distributions to describe both effectors, as they are identical from a dynamics standpoint.

To address the lack of generality of the effector solutions of spinning rigid bodies discussed in Refs. 13 and 11, BSM is expanded in Ref. 14 to develop particular effector solutions for one and two degrees of freedom effectors. The equations of motion of one and two-degree-of-freedom components are derived and implemented in a general manner, avoiding the symmetry assumptions of prior work, to simulate any part mounted on a single or dual gimbal. The result is a general formulation for each effector type (single or dual-axis rotating rigid body), which can simulate reaction wheels, solar panels, control-moment gyroscopes, etc., using only one set of equations for each type. This means that the general equations are validated once and can be used for current and future implementations of one and two-degree-of-freedom components, saving time for both the user and developer.

This work aims to develop a general formulation N-degree-of-freedom sequential rotation rigid components connected to a rigid hub. Depending on the use case, the dynamicist can choose how many degrees of freedom are needed and how they are arranged to simulate a particular component. As an example, a three-degree-of-freedom component type is shown. The formulation is broad enough to include a cluster of three rigid bodies connected sequentially by three rotary joints, a single part able to rotate in a yaw-pitch-roll or any other permutation of three sequential axial rotations with up to three bodies. This dramatically extends the spacecraft configuration space that can be simulated, as three rotations can reach the entire attitude space of a rigid sub-component. Specific examples of these components are found in Figures 2a and 2b. The figure on the left exemplifies a three-panel cluster attached to a hub. As is evident, the hinges on which the panels are deployed do not have to be parallel. The figure on the right shows an example of a two-body component in which the lower arm can rotate about one axis while the upper antenna sits on a dual-axis gimbal. These are just two examples of the numerous components that fit the same dynamic model of three sequential rotations.



(a) Spacecraft with a triple hinged-panel component.



(b) Spacecraft with a dual gimbaled antenna attached to a single-hinged arm.

Figure 2: Three-degree-of-freedom component configurations.

Beyond simulating a chain of rigid bodies, this N-degree-of-freedom formulation can also simulate flexing rigid bodies, first proposed in Ref. 11. It approximates the behavior of a flexible appendage by discretizing it into a series of small rigid bodies connected by joints with springs and dampers. The flexible body modes can be simulated by tuning the spring and damper coefficients. While this is a first-order approximation, it can be helpful in early mission design to test against control-structure interactions by guaranteeing that the control devices do not excite unwanted structural modes.

The present work aims to expand on Ref. 14 and the BSM framework to formulate the generalized dynamics of a sequence of spinning body components. This enables simulating any sequence of rigid body components subject to sequential rotations with respect to the spacecraft's hub. The goal is to determine patterns in the effector differential equations of motion development to allow the sequence of rigid bodies to be written in a summation form. Such a solution retains the ability to describe multiple components that fit the same dynamics type (reaction wheels and single panel for one degree of freedom or control-moment gyroscope and dual panel for two degrees of freedom) while vastly expanding the configuration space by also modeling components with more degrees of freedom. While the rotations considered must be sequential, they can be grouped to form multiple-degree-of-freedom joints. Therefore, the connections between bodies can be represented by universal joints using one-, two- or three-dimensional joints. This is particularly useful for robotic arms with joints between links composed of multiple degrees of freedom. Throughout the equations of motion derivation, no assumption is made on the spin axes, the center of mass locations, the component inertias, etc. Thus, this paper pursues a universal implementation of these dynamic models.

This paper uses Kane's equations,[15,16] as opposed to Newtonian and Eulerian[17,18] formulations employed in prior work.[9–11,13,14,19] While all formulations are equivalent in that they yield analogous equations of motion, Kane's method lends itself better to the N-degree-of-freedom problem. Although more abstract, it can provide more analytical insight into the equations of motion structure. This is especially important because the main goal is to find patterns in the equations that describe an effector with various degrees of freedom. In addition to Kane's equation, this work also makes use of the inertia transport theorem.[20] Analogous to the vector transport theorem,[17] the inertia transport theorem describes the time derivative of the inertia tensor between two different rotating frames. This result is used to simplify the rotational equations of motion to yield a more compact analytical result of the effector differential equations of motion.

This paper is structured as follows. First, a brief introduction to Kane's method is given to familiarize the reader with this dynamics formulation. Second, the problem statement is shown to set up the rigid hub and the N-degree-of-freedom (DoF) effector. Then, all necessary equations of motion are written and derived, which include the system's translation and rotational equations of motion, as well as the equations for each additional degree of freedom of the effector.

KANE'S METHOD

This paper makes extensive use of Kane's method, developed by Kane and Levinson.[15] Ref. 21 further adapts this theory to the back-substitution method and introduces the notation used throughout the paper. While this work is agnostic to the dynamics formulation used in the sense that they all yield analogous equations of motion, Kane's method is beneficial due to its systematic nature in how it develops analytical equations of motion. The main challenge with this work is to find patterns in the accelerations to yield compact and programmatic equations of motion, and Kane's method, while more abstract and arguably less intuitive, is structured in a way that eases the effort of the dynamicist in finding all the necessary equations of motion. Newtonian or Eulerian formulations require understanding where the forces/torques are applied, where to draw the system's bounding box, etc. In contrast, Kane's method only requires that the dynamicist finds the accelerations of each connected body. The resulting equations might need rearrangements before being ready to use with the back-substitution formulation, but this can be done using some algebraic properties.

Kane's method starts by defining the generalized coordinates and their corresponding generalized velocities.[15,16] The velocities do not have to be the direct derivative of the corresponding coordinate. This is particularly important with the rotational coordinates, where any attitude parameterization can be used while utilizing the angular velocity vector as the corresponding generalized velocity.

After defining the generalized coordinate and velocity vectors, the velocity vector of the center of mass and the angular velocity vector of the body's frame are defined. This is because rigid bodies are the focus of this work, so both the translation and rotation properties are investigated. These vectors are used to determine the partial velocities. These are calculated by taking the derivative of each body's velocity and angular velocity vectors with respect to each generalized velocity coordinate.

Finally, the generalized active force F_r and generalized inertia force F_r^* are defined for each degree of freedom. The active force corresponds to external forces/torques. In contrast, the inertia force is computed using the acceleration of the center of mass of each body and the angular acceleration of each corresponding frame. The generalized active force has a translational and a rotational term applied to forces and torques, respectively, and is defined as

$$F_r = \sum_i (\boldsymbol{v}_r^i)^T \cdot \boldsymbol{F} + \sum_i (\boldsymbol{\omega}_r^i)^T \cdot \boldsymbol{L} \tag{1}$$

where $\boldsymbol{v}_r^i$ and $\boldsymbol{\omega}_r^i$ correspond to the translational and rotational partial velocities of the i-th body. Here, the summation is done over all connected bodies. The generalized inertia force also has translational and rotational contributions and is given by

$$F_r^* = -\sum_i (v_r^i)^T \cdot \left(m_{\text{S}_i} \ddot{\boldsymbol{r}}_{S_{c_i}/N}\right) - \sum_i (\omega_r^i)^T \cdot \left([I_{\text{S}_i,S_{c_i}}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S},S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}}\right) \tag{2}$$

where $\ddot{\boldsymbol{r}}_{S_{c_i}/N}$ is the acceleration of the center of mass of the i-th body, while $\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}$ are the angular velocity and angular acceleration of the frame connected to the i-th body, respectively. The mass of the i-th body is m_{S_i} and its inertia about its center of mass is $[I_{\text{S}_i,S_{c_i}}]$.

With the generalized active and inertia forces in hand, the equations of motion are defined by

$$F_r + F_r^* = 0 \tag{3}$$

Depending on the variable r, grouping variables in a vector is sometimes convenient instead of using them component-wise. This is particularly important for the position $\boldsymbol{r}_{B/N}$ and angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$, where three components are used together to form a vector. In this case, both $\boldsymbol{v}_r$ and $\boldsymbol{\omega}_r$ are matrices instead of scalars, represented by $[\boldsymbol{v}_r^{B_c}]$ and $[\boldsymbol{\omega}_r^{\mathcal{B}}]$, and the forces are vectors instead of scalars: $\boldsymbol{F}_r$ for the generalized active force and $\boldsymbol{F}_r^*$ for the generalized inertia force.

EFFECTOR PROBLEM STATEMENT

The problem statement for the N-axes rotating rigid body effector attached to the hub is given in Figure 3. The inertial frame $\mathcal{N}$ originates at point N. The spacecraft is composed of a rigid hub to which the effector

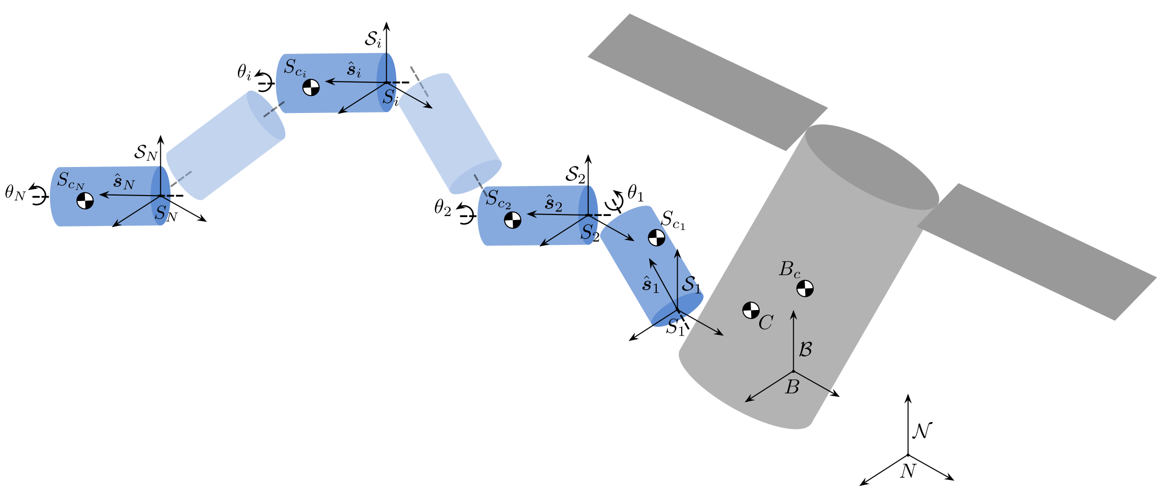


Figure 3: Problem statement for the N-degree-of-freedom spinning rigid body.

attaches. The $\mathcal{B}$ frame has an origin point at B, which does not have to coincide with the center of mass of the hub B_c nor the center of mass of the system C. The hub has mass m_{hub}, center of mass $\boldsymbol{r}_{B_c/B}$ and inertia about its center of mass $[I_{\text{hub},B_c}]$ The effector consists of a chain of rigid bodies connected through single hinges. The mass and inertia of some bodies can be set to zero to give the joint multiple degrees of freedom of an effector component. Here, the degrees of rotation act as an Euler angle sequence of rotation. Each body has a frame $\mathcal{S}_i$ with origin S_i, mass m_{S_i}, center of mass $\boldsymbol{r}_{S_{c_i}/S_i}$ and inertia about its center of mass $[I_{\mathrm{S}_i,S_{c_i}}]$. The bodies rotate the $\hat{\boldsymbol{s}}_i$ spin axis, which passes through the origin of the $\mathcal{S}_\rangle$ frame, with an angle θ_i and angular rate $\dot{\theta}_i$.

Generalized Coordinates

Kane's method requires defining generalized coordinate and velocity vectors containing the system's states and their derivatives. For this problem, the states include the position and attitude of the hub $\boldsymbol{r}_{B/N}$ and $\sigma_{\mathcal{B}/\mathcal{N}}$, respectively, as well as the angles for all joints of the effector. The generalized coordinate vector $\boldsymbol{q}$ and corresponding generalized velocity vector $\boldsymbol{u}$ are

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{r}_{B/N} \\ \boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}} \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} \dot{\boldsymbol{r}}_{B/N} \\ \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix} \tag{4}$$

Velocities

The velocities of the center of mass of all bodies need to be computed to calculate the corresponding translational partial velocities. For the hub, it is

$$\dot{\boldsymbol{r}}_{B_c/N} = \dot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{r}}_{B_c/B} = \dot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{B_c/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{5}$$

where $[\tilde{\boldsymbol{r}}_{B_c/B}]$ corresponds to the cross-product operator such that $[\tilde{\boldsymbol{r}}_{B_c/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{r}_{B_c/B} \times \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$. The transport theorem[17] is used here since $\boldsymbol{r}_{B_c/B}$ is constant in the $\mathcal{B}$ frame. The velocity of the center of mass for the first two rotating bodies is

$$\dot{\boldsymbol{r}}_{S_{c_1}/N} = \dot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{r}}_{S_{c_1}/B} = \dot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_1}/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{r}}_{S_{c_1}/S_1}]\hat{\boldsymbol{s}}_1\dot{\theta}_1 \tag{6}$$

$$\dot{\boldsymbol{r}}_{S_{c_2}/N} = \dot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{r}}_{S_{c_2}/B} = \dot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_2}/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{r}}_{S_{c_2}/S_1}]\hat{\boldsymbol{s}}_1\dot{\theta}_1 - [\tilde{\boldsymbol{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\dot{\theta}_2 \tag{7}$$

A pattern emerges for the velocity of the center of mass of the i-th rotating body

$$\dot{\boldsymbol{r}}_{S_{c_i}/N} = \dot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_j}]\hat{\boldsymbol{s}}_j\dot{\theta}_j \tag{8}$$

This means the translational partial velocities relating to all bodies can be computed.

Angular Velocities

Similarly, the angular velocities of the frames attached to each rigid body are needed. The angular velocities of the first two rotating rigid bodies are

$$\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{s}}_1\dot{\theta}_1 \tag{9}$$

$$\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{s}}_1\dot{\theta}_1 + \hat{\boldsymbol{s}}_2\dot{\theta}_2 \tag{10}$$

Another pattern emerges for the angular velocity of the i-th frame

$$\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{i}\hat{\boldsymbol{s}}_j\dot{\theta}_j \tag{11}$$

Partial Velocity Table

The partial velocities are computed by taking the partial derivative of each body's center of mass velocity $\dot{\boldsymbol{r}}_{S_{c_i}/N}$ or the frame's angular velocity $\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}}$ with respect to each generalized coordinate u_r. Therefore, each entry applies $\partial/\partial r$ to the corresponding variable, where 1-3 is for $\boldsymbol{r}_{B/N}$, 4-6 is for $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and the rest are for all θ_i. The partial velocity table is shown in Table 1.

Table 1: Partial velocities.

r	$\boldsymbol{v}_r^{B_c}$	$\boldsymbol{\omega}_r^{\mathcal{B}}$	$\boldsymbol{v}_r^{S_{c_1}}$	$\boldsymbol{\omega}_r^{\mathcal{S}_1}$	$\boldsymbol{v}_r^{S_{c_2}}$	$\boldsymbol{\omega}_r^{\mathcal{S}_2}$	$\dots$	$\boldsymbol{v}_r^{S_{c_i}}$	$\boldsymbol{\omega}_r^{\mathcal{S}_i}$	$\dots$	$\boldsymbol{v}_r^{S_{c_N}}$	$\boldsymbol{\omega}_r^{\mathcal{S}_N}$
$1-3$	$[I_{3\times3}]$	$[0_{3\times3}]$	$[I_{3\times3}]$	$[0_{3\times3}]$	$[I_{3\times3}]$	$[0_{3\times3}]$	$\dots$	$[I_{3\times3}]$	$[0_{3\times3}]$	$\dots$	$[I_{3\times3}]$	$[0_{3\times3}]$
$4-6$	$-[\tilde{\boldsymbol{r}}_{B_c/B}]$	$[I_{3\times3}]$	$-[\tilde{\boldsymbol{r}}_{S_{c_1}/B}]$	$[I_{3\times3}]$	$-[\tilde{\boldsymbol{r}}_{S_{c_2}/B}]$	$[I_{3\times3}]$	$\dots$	$-[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]$	$[I_{3\times3}]$	$\dots$	$-[\tilde{\boldsymbol{r}}_{S_{c_N}/B}]$	$[I_{3\times3}]$
7	$[0_{3\times3}]$	$[0_{3\times3}]$	$[\tilde{\boldsymbol{r}}_{S_{c_1}/S_1}]\hat{\boldsymbol{s}}_1$	$\hat{\boldsymbol{s}}_1$	$[\tilde{\boldsymbol{r}}_{S_{c_2}/S_1}]\hat{\boldsymbol{s}}_1$	$\hat{\boldsymbol{s}}_1$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_i}/S_1}]\hat{\boldsymbol{s}}_1$	$\hat{\boldsymbol{s}}_1$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_N}/S_1}]\hat{\boldsymbol{s}}_1$	$\hat{\boldsymbol{s}}_1$
8	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[\tilde{\boldsymbol{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2$	$\hat{\boldsymbol{s}}_2$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_i}/S_2}]\hat{\boldsymbol{s}}_2$	$\hat{\boldsymbol{s}}_2$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_N}/S_2}]\hat{\boldsymbol{s}}_2$	$\hat{\boldsymbol{s}}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$6+i$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_i}/S_i}]\hat{\boldsymbol{s}}_i$	$\hat{\boldsymbol{s}}_i$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_N}/S_i}]\hat{\boldsymbol{s}}_i$	$\hat{\boldsymbol{s}}_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$6+N$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$[0_{3\times3}]$	$\dots$	$[0_{3\times3}]$	$[0_{3\times3}]$	$\dots$	$[\tilde{\boldsymbol{r}}_{S_{c_N}/S_N}]\hat{\boldsymbol{s}}_N$	$\hat{\boldsymbol{s}}_N$

TRANSLATIONAL EQUATION OF MOTION

Here, the translational equation of motion for the system consisting of a hub with an N-DoF effector is derived. The goal is to write the equation to fit the form proposed in Ref. 10. For the translation equations of motion, the generalized equation is shown here as

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_{\text{eff}}}\sum_{j=1}^{N_{\text{DOF},i}} \boldsymbol{v}_{\text{Trans,LHS}_{ij}}\ddot{\alpha}_{ij} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \sum_{i=1}^{N_{\text{eff}}}\boldsymbol{v}_{\text{Trans,RHS}_i} \tag{12}$$

There are a few things to note in the equation above. First, all explicit second-order accelerations are moved to the left-hand side. These include the acceleration of the hub, the angular acceleration of the hub's frame, and all second-order time derivatives of the effector's states. Second, while Equation 12 shows a summation of all effectors, this work only derives the equations for one effector, with the result for multiple being analogous. This is a direct consequence of the modularity of the backsubstitution method. Finally, the right-hand side shows some terms using center of mass properties such as $\boldsymbol{c}$ and $\boldsymbol{c}'$, which correspond to the position and velocity (with respect to the $\mathcal{B}$ frame) of the center of mass with respect to point B, respectively. Therefore, the final equation must be rearranged so that these terms are explicit. The remaining terms are grouped into $\boldsymbol{v}_{\text{Trans,RHS}_i}$, which catches any other contributions. The following derivation of the equations of motion takes the form in Equation 12 into account to right the equation into the most compact form suitable for use with the backsubstitution method.

The generalized active force for the system's translation is given by

$$\boldsymbol{F}_{1-3} = \boldsymbol{F}_{\text{ext}} \tag{13}$$

where $\boldsymbol{F}_{\text{ext}}$ is the sum of the external forces on the system. The generalized inertia forces are given by

$$\boldsymbol{F}^*_{1-3} = -\sum_{i=0}^{N} m_{\text{S}_i} (\boldsymbol{v}^{S_{c_i}}_{1-3})^T \cdot \ddot{\boldsymbol{r}}_{S_{c_i}/N} \tag{14}$$

where, for ease of notation, $S_0 \equiv B$ and $S_{c_0} \equiv B_c$, making the hub the 0-th body. No $\boldsymbol{\omega}_r$ terms appear because their respective partial velocities are equal to zero. To find the acceleration of the center of mass of the i-th body, let us first write the acceleration of the first bodies to find a pattern, starting with the hub.

$$\ddot{\boldsymbol{r}}_{B_c/N} = \ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{B_c/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{B_c/B} \tag{15}$$

This result is found by taking the inertial time derivative of Eq. (5) while using the transport theorem to take $\mathcal{B}$ frame derivatives. Applying the same approach to the first and second bodies, their accelerations are given by

$$\begin{aligned}\ddot{\boldsymbol{r}}_{S_{c_1}/N} &= \ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_1}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{r}}_{S_{c_1}/S_1}]\hat{\boldsymbol{s}}_1\ddot{\theta}_1 + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_1}/B} \\ &\quad + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_1}/B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{r}'_{S_{c_1}/S_1}\end{aligned} \tag{16}$$

$$\begin{aligned}\ddot{\boldsymbol{r}}_{S_{c_2}/N} &= \ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_2}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{r}}_{S_{c_2}/S_1}]\hat{\boldsymbol{s}}_1\ddot{\theta}_1 - [\tilde{\boldsymbol{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\ddot{\theta}_2 + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_2}/B} \\ &\quad + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_2}/B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{r}'_{S_{c_2}/S_1} \\ &\quad + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\boldsymbol{r}'_{S_{c_2}/S_2} - [\tilde{\boldsymbol{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}\end{aligned} \tag{17}$$

It is important to note that the transport theorem is used extensively to simplify some derivatives, particularly by using the property that $\boldsymbol{r}_{S_{c_i}/S_i}$ is constant in the $\mathcal{S}_i$ frame and that $\boldsymbol{r}_{S_i/S_{i-1}}$ is constant in the $\mathcal{S}_{i-1}$ frame. For the i-th body, the following pattern emerges

$$\begin{aligned}\ddot{\boldsymbol{r}}_{S_{c_i}/N} =& \ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_j}]\hat{\boldsymbol{s}}_j\ddot{\theta}_j + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_i}/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_i}/B} \\ & + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\end{aligned} \tag{18}$$

Using Kane's Method, the translational equation of motion is given by setting $\boldsymbol{F}_{1-3} + \boldsymbol{F}^*_{1-3} = \boldsymbol{0}$, which yields

$$\begin{aligned}&\sum_{i=0}^{N} m_{\text{S}_i}\ddot{\boldsymbol{r}}_{B/N} - \sum_{i=0}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N} m_{\text{S}_i}\sum_{j=1}^{i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_j}]\hat{\boldsymbol{s}}_j\ddot{\theta}_j = \\ &= \boldsymbol{F}_{\text{ext}} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\sum_{i=1}^{N} m_{\text{S}_i}\boldsymbol{r}'_{S_{c_i}/B} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\sum_{i=0}^{N} m_{\text{S}_i}\boldsymbol{r}_{S_{c_i}/B} \\ &+ \sum_{i=1}^{N} m_{\text{S}_i}\left(\sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\right)\end{aligned} \tag{19}$$

Note here that the explicit second-order acceleration terms are grouped to the left-hand side in the equation above, and many terms are already written in a suitable form according to Ref. 10, particularly one that allows the use of the system's center of mass properties. It is useful to define the following mass property terms

$$m_{\text{sc}} = \sum_{i=0}^{N} m_{\text{S}_i} \tag{20}$$

$$m_{\text{sc}}\boldsymbol{c} = \sum_{i=0}^{N} m_{\text{S}_i}\boldsymbol{r}_{S_{c_i}/B} \tag{21}$$

$$m_{\text{sc}}\boldsymbol{c}' = \sum_{i=0}^{N} m_{\text{S}_i}\boldsymbol{r}'_{S_{c_i}/B} \tag{22}$$

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}] = \sum_{i=0}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}] \tag{23}$$

In addition, because it is convenient to have a single summation in the $\hat{\boldsymbol{s}}_i\ddot{\theta}_i$ terms, the following property of double summations is used

$$\sum_{i=1}^{N}\sum_{j=1}^{i} f_{ij} = \sum_{i=1}^{N}\sum_{j=i}^{N} f_{ji} \tag{24}$$

Finally, the compact form of the translational equation of motion is given by

$$\begin{aligned} m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}\left(\sum_{j=i}^{N} m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\right)\hat{\boldsymbol{s}}_i\ddot{\theta}_i = \boldsymbol{F}_{\text{ext}} \\ - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \\ + \sum_{i=1}^{N} m_{\text{S}_i}\left(\sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\right) \end{aligned} \tag{25}$$

ROTATIONAL EQUATION OF MOTION

The rotational equation of motion defines the attitude history of the system. Again, the goal is to write it as presented in Ref. 10, which shows that the generalized rotational equation takes the form

$$\begin{aligned} m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_{\text{eff}}}\sum_{j=1}^{N_{\text{DOF},i}} \boldsymbol{v}_{\text{Rot,LHS}_{ij}}\ddot{\alpha}_{ij} = \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_{\text{eff}}} \boldsymbol{v}_{\text{Rot,RHS}_i} \end{aligned} \tag{26}$$

As before, the explicit second-order terms are moved to the left-hand side. The right-hand side contains terms using the systems mass properties, which now include the system's total inertia $[I_{\text{sc},B}]$ and its $\mathcal{B}$-frame derivative $[I_{\text{sc},B}]'$, as well as the catch-all term $\boldsymbol{v}_{\text{Rot,RHS}_i}$ for all other terms.

For the rotational equation of motion, the generalized active force corresponds to the applied torque $\boldsymbol{F}_{4-6} = \boldsymbol{L}_B$, where $\boldsymbol{L}_B$ is the pure torque applied to the origin of the $\mathcal{B}$ frame. The generalized inertia force is

$$\boldsymbol{F}^*_{4-6} = -\sum_{i=0}^{N}\left(m_{\text{S}_i}(\boldsymbol{v}^{S_{c_i}}_{4-6})^T\ddot{\boldsymbol{r}}_{S_{c_i}/N} + (\boldsymbol{\omega}^{\mathcal{S}_i}_{4-6})^T\left([I_{\text{S}_i,S_{c_i}}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}}\right)\right) \tag{27}$$

It is important to note that none of the partial velocities are zero here, which means all terms are accounted for. To find the angular acceleration of each frame, let us write the first two results

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{s}}_1\ddot{\theta}_1 - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{28}$$

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{s}}_1\ddot{\theta}_1 + \hat{\boldsymbol{s}}_2\ddot{\theta}_2 - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \tag{29}$$

These results are obtained by applying the transport theorem, which makes use of the fact that $\hat{\boldsymbol{s}}_i$ is constant in both the $\mathcal{S}_i$ and $\mathcal{S}_{i-1}$ frames. Again, a pattern emerges for the i-th frame

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{N}\hat{\boldsymbol{s}}_j\ddot{\theta}_j - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}} \tag{30}$$

Kane's equation $\boldsymbol{F}_{4-6} + \boldsymbol{F}^*_{4-6} = \boldsymbol{0}$ can now be used

$$\boldsymbol{L}_B - \sum_{i=0}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\ddot{\boldsymbol{r}}_{S_{c_i}/N} - \sum_{i=0}^{N}[I_{\text{S}_i,S_{c_i}}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} - \sum_{i=0}^{N}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} = \boldsymbol{0} \tag{31}$$

To simplify the equation above, the last term is expanded

$$\begin{aligned}\sum_{i=0}^{N}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} &= [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(\sum_{i=0}^{N}[I_{\text{S}_i,S_{c_i}}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \left(\sum_{i=0}^{N}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}][I_{\text{S}_i,S_{c_i}}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &\quad + \sum_{i=0}^{N}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{B}}\end{aligned} \tag{32}$$

Moreover, the triple cross product identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} + \boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})$ is used to get the following equalities

$$[\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}] = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \tag{33}$$

$$[\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_i}/B} = -[\tilde{\boldsymbol{r}}'_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{r}'_{S_{c_i}/B} \tag{34}$$

These are used to group like-terms together and to fit the intended final form of the equation. Expanding the acceleration terms and plugging these results in Eq. (31) yields

$$\begin{aligned}&\sum_{i=0}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\ddot{\boldsymbol{r}}_{B/N} + \sum_{i=0}^{N}\Big([I_{\text{S}_i,S_{c_i}}] - m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{i}\Big([I_{\text{S}_i,S_{c_i}}] - m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/S_j}]\Big)\hat{\boldsymbol{s}}_j\ddot{\theta}_j = \\ &= \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\sum_{i=0}^{N}\Big([I_{\text{S}_i,S_{c_i}}] - m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &- \sum_{i=0}^{N}\Big([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}][I_{\text{S}_i,S_{c_i}}] - [I_{\text{S}_i,S_{c_i}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}] - m_{\text{S}_i}\Big([\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}'_{S_{c_i}/B}] + [\tilde{\boldsymbol{r}}'_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big)\Big)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &- \sum_{i=1}^{N}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{B}} - [I_{\text{S}_i,S_{c_i}}]\sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}}\right) \\ &- \sum_{i=1}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\left([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{r}'_{S_{c_i}/B} + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j}\right. \\ &\left. - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\right)\end{aligned} \tag{35}$$

Once again, all explicit second-order acceleration terms are grouped on the left-hand side. The following system inertia definitions are useful

$$[I_{\text{sc},B}] = \sum_{i=0}^{N} \left([I_{\text{S}_i,S_{c_i}}] - m_{\text{S}_i} [\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}] \right) \tag{36}$$

$$[I'_{\text{sc},B}] = \sum_{i=0}^{N} \left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}][I_{\text{S}_i,S_{c_i}}] - [I_{\text{S}_i,S_{c_i}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}] - m_{\text{S}_i} \left([\tilde{\boldsymbol{r}}_{S_{c_i}/B}][\tilde{\boldsymbol{r}}'_{S_{c_i}/B}] + [\tilde{\boldsymbol{r}}'_{S_{c_i}/B}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}] \right) \right) \tag{37}$$

The equation of motion above can be simplified to show the compact form of the rotational equation of motion

$$\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N} \left(\sum_{j=i}^{N} \left([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j} [\tilde{\boldsymbol{r}}_{S_{c_j}/B}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}] \right) \right) \hat{\boldsymbol{s}}_i \ddot{\theta}_i = \\
= \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
- \sum_{i=1}^{N} \left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{B}} - [I_{\text{S}_i,S_{c_i}}] \sum_{j=1}^{i-1} [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}} \right) \\
- \sum_{i=1}^{N} m_{\text{S}_i} [\tilde{\boldsymbol{r}}_{S_{c_i}/B}] \Bigg([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{r}'_{S_{c_i}/B} + \sum_{j=1}^{i} [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} \\
- \sum_{j=1}^{i-1} [\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j} \Bigg)
\end{aligned} \tag{38}$$

N-TH SPINNING BODY EQUATION OF MOTION

From Ref. 10, for a single uncoupled degree of freedom, the governing equation of motion takes the form

$$\ddot{\alpha}_i = \boldsymbol{a}_{\alpha_i} \cdot \ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{b}_{\alpha_i} \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\alpha_i} \tag{39}$$

where α_i is the state and $\ddot{\alpha}_i$ the corresponding second-order time derivative. The left-hand side is only taken by the states' acceleration, while the right-hand side contains three terms: $\ddot{\boldsymbol{r}}_{B/N}$, $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and the last term, which conglomerates all remaining contributions. The equations are written this way because this result for the state acceleration is "backsubstituted" into the translational and rotational equations, which means $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ can be solved for independently of any $\ddot{\alpha}_i$. The result is then substituted in Equation 39, and all states can be numerically propagated forward in time because their accelerations have been explicitly solved.

However, many effectors have multiple coupled degrees of freedom, meaning the equations do not fit into the form presented in Equation 39. For these cases, it is useful to write the equations in a coupled form using a mass matrix, as shown below

$$[\boldsymbol{M}_\alpha]\ddot{\boldsymbol{\alpha}} = [\boldsymbol{A}^*_\alpha]\ddot{\boldsymbol{r}}_{B/N} + [\boldsymbol{B}^*_\alpha]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{C}^*_\alpha] \tag{40}$$

where $[\boldsymbol{M}_\alpha]$ is the mass matrix and $\ddot{\boldsymbol{\alpha}}$ is the stacked vector of states for the effector. In some cases, particularly when the number of degrees of freedom is small, the matrix equation in Eq. 40 can be analytically solved for, yielding several equations like Eq. 39. However, this is not always possible and can be tedious, so a faster option is to invert the mass matrix $[\boldsymbol{M}_\alpha]$. As with the global mass matrix, this can be computationally expensive, but it is a viable solution given that this is done for each effector, not the entire system. Using this approach, the canonical form of equation (40) is given by

$$\ddot{\boldsymbol{\alpha}} = [\boldsymbol{A}_\alpha]\ddot{\boldsymbol{r}}_{B/N} + [\boldsymbol{B}_\alpha]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{C}_\alpha] \tag{41}$$

where the new matrices are defined as

$$[\boldsymbol{A}_\alpha] = [\boldsymbol{M}_\alpha]^{-1}[\boldsymbol{A}^*_\alpha], \qquad [\boldsymbol{B}_\alpha] = [\boldsymbol{M}_\alpha]^{-1}[\boldsymbol{B}^*_\alpha], \qquad [\boldsymbol{C}_\alpha] = [\boldsymbol{M}_\alpha]^{-1}[\boldsymbol{C}^*_\alpha] \tag{42}$$

This section aims to find the equation for each row of the matrix equation in Eq. 40. For the n-th equation of motion, the generalized active force is $F_n = u_n$ and the generalized inertia force is

$$F^*_n = -\sum_{i=n}^{N}\Big(m_{\mathrm{S}_i}[\boldsymbol{v}_n^{S_{c_i}}]^T\ddot{\boldsymbol{r}}_{S_{c_i}/N} + [\boldsymbol{\omega}_n^{\mathcal{S}_i}]^T\left([I_{\mathrm{S}_i,S_{c_i}}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\mathrm{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}}\right)\Big) \tag{43}$$

where, using the partial velocity table, $[\boldsymbol{v}_n^{S_{c_i}}] = [\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\hat{\boldsymbol{s}}_n$ and $[\boldsymbol{\omega}_n^{\mathcal{S}_i}] = \hat{\boldsymbol{s}}_n$.

Kane's equation $F_n + F^*_n = 0$ is used to get

$$\begin{aligned}
&\hat{\mathbf{s}}_n^T\sum_{i=n}^{N}\sum_{j=1}^{i}\Big([I_{\mathrm{S}_i,S_{c_i}}] - m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_i}/S_j}]\Big)\hat{\boldsymbol{s}}_j\ddot{\theta}_j = u_n \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N} m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\ddot{\boldsymbol{r}}_{B/N} - \hat{\mathbf{s}}_n^T\sum_{i=n}^{N}\Big([I_{\mathrm{S}_i,S_{c_i}}] - m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\mathrm{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} - [I_{\mathrm{S}_i,S_{c_i}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}}\right) \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N} m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\Big([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_i}/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_i}/B} \\
&\quad + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\Big)
\end{aligned} \tag{44}$$

Notice how the state acceleration is moved to the left while all other terms are on the right. Moreover, the $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ are explicitly written, and every other term consists of additional contributions that do not explicitly depend on state accelerations. As usual, it is convenient to rewrite the double summation in $\hat{\boldsymbol{s}}_j\ddot{\theta}_j$ using Eq. 24, which yields the alternative form

$$\begin{aligned}
&\hat{\mathbf{s}}_n^T\sum_{i=1}^{n}\left(\sum_{j=n}^{N}\Big([I_{\mathrm{S}_j,S_{c_j}}] - m_{\mathrm{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\right)\hat{\boldsymbol{s}}_i\ddot{\theta}_i \\
&+\hat{\mathbf{s}}_n^T\sum_{i=n+1}^{N}\left(\sum_{j=i}^{N}\Big([I_{\mathrm{S}_j,S_{c_j}}] - m_{\mathrm{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\right)\hat{\boldsymbol{s}}_i\ddot{\theta}_i = u_n \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N} m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\ddot{\boldsymbol{r}}_{B/N} - \hat{\mathbf{s}}_n^T\sum_{i=n}^{N}\Big([I_{\mathrm{S}_i,S_{c_i}}] - m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\mathrm{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} - [I_{\mathrm{S}_i,S_{c_i}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\mathrm{S}_i,S_{c_i}}]\sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}}\right) \\
&-\hat{\mathbf{s}}_n^T\sum_{i=n}^{N} m_{\mathrm{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\Big([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_i}/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_i}/B} \\
&\quad + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\Big)
\end{aligned} \tag{45}$$

BACKSUBSTITUTION FORMULATION

The backsubstitution method requires taking Equation 40 and plugging the result into Equations 12 and 26, which removes the explicit dependency of the effector second-order state derivatives from the translational and rotational equations. The result is a 6x6 system of equations in $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ given by

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{\text{Trans}} \\ \boldsymbol{v}_{\text{Rot}} \end{bmatrix} \tag{46}$$

Here, $\boldsymbol{v}_{\text{Trans}}$ and $\boldsymbol{v}_{\text{Rot}}$ correspond to all the terms that do not explicitly depend on any second-order time derivatives in the translational and rotational equations of motion, respectively. As discussed in Ref. 10, this system of 6 equations is solved using the Schur complement matrix formulation for the partitioned form of the hub system mass matrix:

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \Big([D] - [C][A]^{-1}[B]\Big)^{-1}(\boldsymbol{v}_{\text{Rot}} - [C][A]^{-1}\boldsymbol{v}_{\text{Trans}}) \tag{47}$$

$$\ddot{\boldsymbol{r}}_{B/N} = [A]^{-1}(\boldsymbol{v}_{\text{Trans}} - [B]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}) \tag{48}$$

Let us define each term introduced in the equations above for the problem at hand. First, the terms in Equation 40 are defined. The element of the mass matrix in the n-th row and i-th column is defined as

$$\boldsymbol{M}_{\alpha_{n,i}} = \begin{cases} \hat{\boldsymbol{s}}_n^T \sum_{j=n}^{N} \Big([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\hat{\boldsymbol{s}}_i, & i \le n \\ \hat{\boldsymbol{s}}_n^T \sum_{j=i}^{N} \Big([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\hat{\boldsymbol{s}}_i, & i > n \end{cases} \tag{49}$$

The other terms are defined n-th row-wise as

$$\boldsymbol{A}^*_{\alpha_n} = -\hat{\boldsymbol{s}}_n^T \sum_{i=n}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}] \tag{50}$$

$$\boldsymbol{B}^*_{\alpha_n} = -\hat{\boldsymbol{s}}_n^T \sum_{i=n}^{N} \Big([I_{\text{S}_i,S_{c_i}}] - m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Big) \tag{51}$$

$$\begin{aligned} \boldsymbol{C}^*_{\alpha_n} = u_n - \hat{\boldsymbol{s}}_n^T \sum_{i=n}^{N} \Bigg(& [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} - [I_{\text{S}_i,S_{c_i}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ & -[I_{\text{S}_i,S_{c_i}}]\sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}} + m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_n}]\Big([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_{c_i}/B} \\ & +2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_{c_i}/B} + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\Big)\Bigg) \end{aligned} \tag{52}$$

With the effector equation terms defined, the terms for solving the translational and rotation acceleration can be found. The matrices for this problem are given by

$$[A] = m_{\text{sc}}[I_{3\times3}] - \sum_{i=1}^{N}\left(\sum_{j=i}^{N} m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\right)\hat{\boldsymbol{s}}_i\boldsymbol{A}_{\alpha_i} \tag{53}$$

$$[B] = -m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{i=1}^{N}\left(\sum_{j=i}^{N} m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\right)\hat{\boldsymbol{s}}_i\boldsymbol{B}_{\alpha_i} \tag{54}$$

$$[C] = m_{\text{sc}}[\tilde{\mathbf{c}}] + \sum_{i=1}^{N}\left(\sum_{j=i}^{N}\Big([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/B}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\right)\hat{\boldsymbol{s}}_i\boldsymbol{A}_{\alpha_i} \tag{55}$$

$$[D] = [I_{\text{sc},B}] + \sum_{i=1}^{N}\left(\sum_{j=i}^{N}\Big([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/B}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\right)\hat{\boldsymbol{s}}_i\boldsymbol{B}_{\alpha_i} \tag{56}$$

Notice how the $\boldsymbol{A}_{\alpha_i}$ and $\boldsymbol{B}_{\alpha_i}$ terms are used instead of $\boldsymbol{A}^*_{\alpha_i}$ and $\boldsymbol{B}^*_{\alpha_i}$. The latter are defined directly from each spinning body equation but cannot be used directly. The effector mass matrix must be inverted, resulting in the former terms that cannot be defined analytically. The additional translational and rotational contributions are

$$\begin{aligned}
\boldsymbol{v}_{\text{Trans}} &= \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \\
&+ \sum_{i=1}^{N} m_{\text{S}_i}\left(\sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} - \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\right) \\
&+ \sum_{i=1}^{N}\left(\sum_{j=i}^{N} m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\right)\hat{\boldsymbol{s}}_i\boldsymbol{C}_{\alpha_i}
\end{aligned} \tag{57}$$

$$\begin{aligned}
\boldsymbol{v}_{\text{Rot}} &= \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&- \sum_{i=1}^{N}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\text{S}_i,S_{c_i}}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{B}} - [I_{\text{S}_i,S_{c_i}}]\sum_{j=1}^{i-1}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{S}_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{S}_{j-1}}\right) \\
&- \sum_{i=1}^{N} m_{\text{S}_i}[\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\Bigg([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_{c_i}/B}]\boldsymbol{r}'_{S_{c_i}/B} + \sum_{j=1}^{i}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{S}_{j-1}}]\boldsymbol{r}'_{S_{c_i}/S_j} \\
&- \sum_{j=1}^{i-1}[\tilde{\boldsymbol{r}}_{S_{c_i}/S_{j+1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_{j+1}/\mathcal{S}_j}\Bigg) \\
&- \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\Big([I_{\text{S}_j,S_{c_j}}] - m_{\text{S}_j}[\tilde{\boldsymbol{r}}_{S_{c_j}/B}][\tilde{\boldsymbol{r}}_{S_{c_j}/S_i}]\Big)\right)\hat{\boldsymbol{s}}_i\boldsymbol{C}_{\alpha_i}
\end{aligned} \tag{58}$$

VERIFICATION

A fundamental step in any simulation concerns the verification of the equations developed. Since the equations of motion represent a physical system, conservation laws can be used to verify that both the software implementation and the underlying equations are correct. This is impossible if some cross-coupling terms, even the smaller ones, are removed. In particular, the principles of energy and angular momentum conservation can be used to verify that the model behaves according to physical laws. In an environment where only conservative forces and torques are present, the energy and angular momentum of the entire system must be conserved. In other words, the energy (a scalar) and the angular momentum, expressed in the inertial frame $\mathcal{N}$, (a vector), must not change throughout the simulation. Furthermore, energy and angular momentum can be separated into their orbital and rotational contributions, which must remain constant. This separation is done due to their large order of magnitude difference; the orbital part tends to be much greater than the rotational part. Not separating the two contributions could result in minor errors in one of the rotational components not being caught, as the orbital term overshadows the result.

In line with the back-substitution method, the energy and angular momentum equations are written to separate the hub and center of mass properties from the specific effector contributions. Here, only the effector contributions to the rotational terms are shown; see Ref. 10 for details on the full equations. The contribution to the rotational energy is

$$T_{\text{rot}} = \sum_{i=1}^{N} \left(\frac{1}{2} \boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} \cdot [I_{\text{S}_i,S_{c_i}}] \boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} + \frac{1}{2} m_{\text{S}_i} \dot{\boldsymbol{r}}_{S_{c_i}/B} \cdot \dot{\boldsymbol{r}}_{S_{c_i}/B} \right) \tag{59}$$

while the contribution to the rotational angular momentum is

$$\boldsymbol{H}_{\text{rot},C} = \sum_{i=1}^{N} \left([I_{\text{S}_i,S_{c_i}}] \boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} + m_{\text{S}_i} \boldsymbol{r}_{S_{c_i}/B} \times \dot{\boldsymbol{r}}_{S_{c_i}/B} \right) \tag{60}$$

The conservation laws are verified by a simulation that includes the effectors modeled in this paper. The energy and angular momentum at each timestep are compared to their values at $t = 0$. This comparison is done by taking the difference between the two values and dividing it by the initial value, which yields a relative difference. Due to the finite number of bits in computers, the relative difference is never zero, but rather very close to machine precision, on the order of 10^{-15}. Moreover, the relative difference does not stay completely constant. Instead, it presents as a random walk, common when using fixed timestep integrators. See Ref. 12, 14 for example figures.

CONCLUSION

It is clear that space vehicles are becoming more complex, and it is more important than ever that extensive simulation work is done before the mission launches. Many spacecraft have a rigid hub with attached appendages, many of them comprised of rotating rigid bodies. While these include simple devices like reaction wheels and single-hinged panels, interest has been growing in multi-link robotic arms in recent years. These effectors are useful in servicing missions, but they are hard to model correctly, which this paper achieves in a general way.

The equations of motion of components with N degrees of freedom are thoroughly derived, including the system's translational and rotational equations and an equation for each additional degree of freedom. The derivation is done using the backsubstitution method, which creates a modular formulation to propagate the motion equations that prevent inverting large mass matrices.

Future work includes implementing these equations into software and validating the results using energy and momentum verifications.

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