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# ROTATING RIGID BODY DYNAMICS ARCHITECTURE FOR SPACECRAFT SIMULATION SOFTWARE IMPLEMENTATION

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Deriving and propagating the equations of motion of a spacecraft is fundamental to describing its behavior accurately. These equations of motion depend on the spacecraft's configuration, which includes any physical subsystem components such as attitude control devices, solar panels, gimbals, etc. While the contribution of each subsystem on the space vehicle can be defined independently, most work has focused on deriving them specifically for each component type. This lack of generality yields different formulations for components that are identical from a dynamics standpoint. This paper relaxes the assumptions made in deriving some subsystem components, yielding a general architecture that uses common equations of motion for components representing the same physical reality. The result is a redesigned dynamics architecture along with a set of general equations of motion common to subsystem components where specific assumptions can be applied to describe particular components. This general formulation saves on validation effort and allows for a common software description where inheritance can define specific components.

## INTRODUCTION

Spacecraft simulations are a critical part of any mission, from CubeSats to deep space missions. They allow for detailed analysis of the spacecraft's dynamics, ultimately informing how it will behave and if the mission requirements are met. As missions become more complex, so do the simulations for the spacecraft's behavior. For example, whereas many spacecraft use rigid solar panels, new missions like the Lucy mission to the Trojan asteroids have started to use flexible solar panels[1] to meet higher power needs. Another example of this increased complexity relates to the main thruster platform. While many spacecraft attach the thruster directly to the system's hub, some have opted to use a gimbaled platform instead. This is particularly useful for spacecraft using ionic thrusters,[2] as they tend to thrust for long periods and need to account for offsets between the thrust vector and the center of mass. Missions like Deep Space 1,[3] Dawn,[4] and Psyche[5] all use this technology. While both these features are mission-critical, they add a layer of complexity that needs to be included in spacecraft simulations.

One of the critical steps for these comprehensive, high-fidelity simulations is the derivation of the spacecraft's equations of motion. By numerically propagating them, the behavior of the spacecraft

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and its components can be described and analyzed. The equations of motion also need to respect physical conservation laws, which are used to validate them. Another important aspect is the software implementation of these equations. A modular, general software architecture allows for faster prototyping and guarantees the model's fidelity with increasing complexity.

Previous work has focused on the derivation of equations of motion in a modular way, separating each component's contributions by assuming that each is connected directly to a common spacecraft's hub.[6] Some examples include reaction wheels,[7] variable-speed control moment gyroscopes[8] and solar arrays.[9] The main drawback of this past work is that the equations of motion are specific to each component, even when different components represent the same reality from a dynamics standpoint. This means that the equations of motion have to be derived, implemented, and validated for each element. Moreover, while the problem of creating the architecture that supports these simulations has been solved in a modular way, the software implementation also relies on specifying the type of component in the model. The lack of a general formulation and implementation of the equations of motion means hidden repeated code, which translates into more work for the developer and potential errors or bugs.

This paper aims to take a more general approach by deriving and implementing the general equations of motion for rotating body components with one or two degrees of freedom. Rotating bodies with one degree of freedom include reaction wheels and single-hinged solar panels, while control moment gyroscopes and dual-gimbaled thrusters are considered two-degree-of-freedom components. The derivations begin by considering the entire system to develop the translational and rotational equations of motion of the system's hub. The components are then considered separately, leading to their rotational motion equation. A general software architecture is proposed, which allows for specific components to inherit from a parent class with shared equations of motion along with common quantities such as mass, inertia, and rotation axis.

To find the general equations of motion without making any assumptions on the rotating rigid bodies, the inertia tensor transport theorem is applied extensively.[10] This theorem converts the time derivative of the inertia tensor in one frame to another frame, analogous to the vector transport theorem.[11] With this theorem, no assumptions on the frame are needed to derive the spacecraft's equations of motion. This is the crucial aspect to allow for a general formulation of the equations that describe classes of rotating rigid bodies.

The outcome of this work is a general analytical description and software implementation of these rotating bodies that is agnostic to the type of rotating body being simulated. These results can be applied to various scenarios, and the proposed architecture can be implemented in any software package. The complete analytical derivation of the equations of motion is shown in this paper, and the software implementation is done in Basilisk,[12] an open-source*, spacecraft-centric simulation software. It has a modular implementation of spacecraft dynamics and flight software modules and has seen extensive use in mission analysis. This paper is organized as follows. First, the generalized dynamics architecture is defined, which covers how different spacecraft components can be grouped into similar classes. Then, the problem statement, equations of motion, and numerical validation for the one-degree-of-freedom system are shown. This process is then repeated for the two-degree-of-freedom system.

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*http://hanspeterschaub.info/basilisk/index.html

## GENERALIZED DYNAMICS ARCHITECTURE

The goal of the proposed dynamics is to uniformize and unify the description of spacecraft components representing identical physical elements from a dynamics point of view. This is done by creating common classes for single and dual-axis rotating rigid bodies that make no assumptions on frame definitions, the center of mass location, inertia distribution, etc. The specific components can then inherit from this class and add other properties that distinguish them from other elements.

A diagram for the single-axis rotating rigid body class is shown in Figure 1 as a gold box. It represents the skeleton for rigid spacecraft appendages that rotate about one axis. The class contains several variables common to all single-axis rotating rigid bodies, like the body's mass and inertia matrix. The center of mass location and spin axis can be defined as any vectors without any assumptions on how the frame is defined. The frame conversion information relating the rotating rigid body frame to the body-fixed frame is expressed through a direction cosine matrix (DCM). The class also contains methods like the equations of motion that describe the system, the mass property contributions to compute the spacecraft's center of mass and inertia matrix, and the energy and momentum contributions to calculate the total energy and angular momentum of the spacecraft.

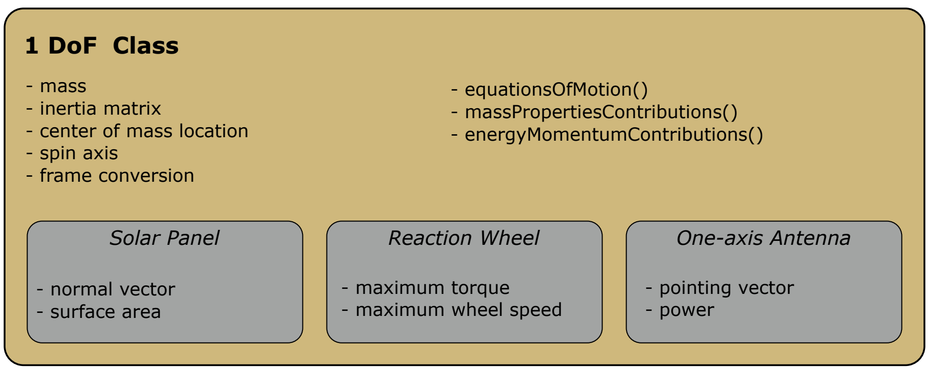


**Figure 1. Class diagram for the one-axis rotating rigid body class.**

The light grey boxes represent specific modules that can be derived from the general one-degree-of-freedom class. These include hinged solar panels, reaction wheels, or one-axis gimbaled antennas. These modules inherit from the 1 DoF class, which means they all contain the same variables and methods from the parent class. However, each module can be specified by adding new variables and methods that define that particular component type. For example, a solar panel needs a vector normal to the solar cells to point at the Sun, as well as the total surface area of the solar cells. Adding additional parameters makes the module more specific while retaining the variables and methods common to all single-axis rotating rigid bodies.

A diagram for the dual-axis rotating rigid body class is shown in Figure 2. This class contains similar variables and methods but is now adapted to represent a two-degree-of-freedom system. There are now two masses, inertia matrices, center of mass locations, spin axes, and DCM frame conversions. The methods have identical names but are adapted to dual-axis kinematics and dynamics.

Similar to the 1 DoF class, modules represented in light grey inherit variables and functions from the 2 DoF parent class. Examples include dual-hinged solar panels, control moment gyroscopes,

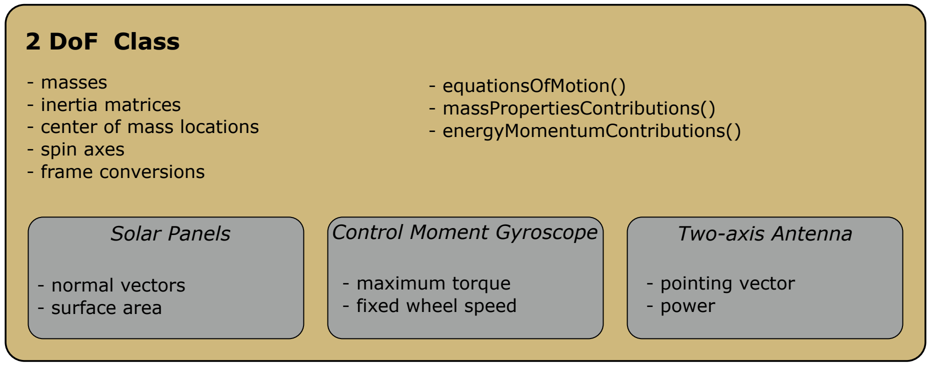


**Figure 2. Class diagram for the two-axis rotating rigid body class.**

and two-axis gimbaled antennas. Apart from being derived from the parent class, each module contains variables and methods that define the specific component, such as a pointing vector for the antenna or a surface area for the solar panels.

Neither parent class is accessible from the user's perspective. Instead, the user instantiates the specific modules, populates the necessary variables that define the particular component, and uses the available methods. The abstract parent class creates a skeleton common to either one or two-degree-of-freedom systems, with the module inheriting its variables and methods. This minimizes repeated code and potential errors or bugs from going over similar processes multiple times.

Moreover, these general classes facilitate the validation process. Instead of having to validate all functions for each module, the common methods such as `equationsOfMotion()`, `massPropertiesContributions()` and `energyMomentumContributions()` can be validated once for the entire parent class. This saves time for the developer, as all shared functions are derived and implemented only once. It also decreases the total number of unit tests, improving the compactness of the software package since common methods are validated only once.

## SINGLE-AXIS ROTATING RIGID BODY DYNAMICS

Here, the derivation and validation of the equations of motion of a single-axis rotating rigid body attached to a rigid hub are shown. The one-degree-of-freedom component can be described as a rigid body that can only rotate about one body-fixed axis. It represents a rigid component attached to the spacecraft's hub through a rotary joint.

This general description can describe multiple common spacecraft components. Examples include single-hinge solar arrays for deployment or first-order flexing analysis, reaction wheels as attitude control devices, and one-axis gimbaled low-gain antennas. All these components can be defined through a general description where they are specified by their mass, inertia matrix, the location of the center of mass, and spin axis.

### Problem Statement

The problem statement for the single-axis rotating rigid body is given in Figure 3. The inertial frame is represented by $\mathcal{N}$ with origin at point $N$. The spacecraft is composed of a rigid body connected to a rigid hub through a single axis of rotation. The hub has a body-fixed frame $\mathcal{B}$ with origin $B$, and its center of mass is located at point $B_c$. The mass of the hub is $m_{\text{hub}}$, and its inertia

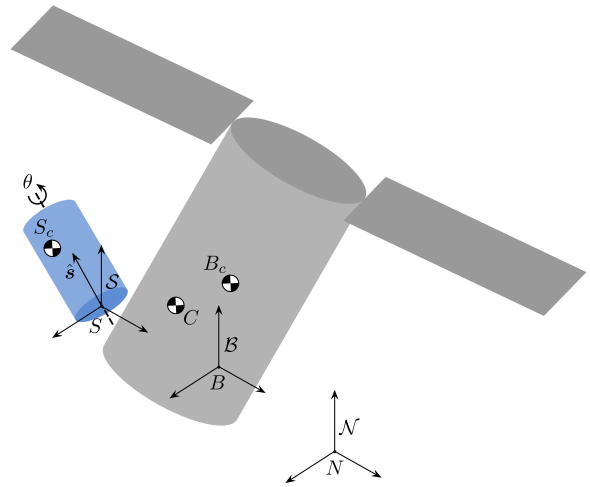


**Figure 3. Problem statement for the one-degree-of-freedom spinning rigid body.**

matrix about point $B$ is $[I_{\text{hub},B}]$. The rotating rigid body has the $\mathcal{S}$ frame attached to it with its origin at point $S$. The center of mass of the spinner is located at point $S_c$. The mass of the spinner is $m_{\text{S}}$, and its inertia matrix about its center of mass is $[I_{\text{S},S_c}]$. The combined center of mass of the system is located at point $C$. The spin axis $\hat{\boldsymbol{s}}$ is constant, as seen by the $\mathcal{B}$ frame, and passes through the point $S$. The angle about the rotation axis is $\theta$, and its angle rate is $\dot{\theta}$.

The single-axis rotating rigid body attached to the hub has seven degrees of freedom shown in Table 1: three for the system's position, three for the system's attitude, and one for the angle about the rotation axis. The motion equations are developed so that all degrees of freedom are described. The position state variables are described by the translational equation of motion, the attitude state variables by the rotational equation of motion, and the rotation angle by the spinner equation of motion.

| State Variables | Degrees of Freedom | Equations of motion |
|---|---|---|
| $\mathbf{r}_{B/N}, \dot{\mathbf{r}}_{B/N}$ | 3 | Translational |
| $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}, \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ | 3 | Rotational |
| $\theta, \dot{\theta}$ | 1 | Spinner Rotational |

**Table 1. State variables for the single-axis rotating rigid body spacecraft.**

### Translational Equations of Motion

The entire system is considered for the translational equation of motion, including the hub and the spinner. This equation of motion defines three degrees of freedom of the system. Using the Super Particle Theorem:

$$m_{\text{sc}}\ddot{\mathbf{r}}_{C/N} = m_{\text{sc}}\ddot{\mathbf{r}}_{B/N} + m_{\text{sc}}\ddot{\mathbf{c}} = \mathbf{F} \tag{1}$$

where $\mathbf{c} \equiv \mathbf{r}_{C/B}$ is the vector from the origin of the body frame $B$ to the system's center of mass $C$, and $\mathbf{F}$ is the combined force acting on the system. A single dot above a vector represent the first-order inertial frame derivative, and a double dot represents the second-order inertial frame derivative. Using the definition of the center of mass of the system:

$$m_{\text{sc}}\mathbf{c} = m_{\text{hub}}\mathbf{r}_{B_c/B} + m_{\text{S}}\mathbf{r}_{S_c/B} \tag{2}$$

Using the transport theorem, the inertial time derivative is expressed using body-frame derivatives as

$$\dot{\mathbf{c}} = \mathbf{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c} \tag{3}$$
$$\ddot{\mathbf{c}} = \mathbf{c}'' + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\mathbf{c}} \tag{4}$$

where a single apostrophe represents a first-order body-frame derivative and a double apostrophe represents a second-order body-frame derivative. The term $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ represents the angular velocity of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame. As for the body-frame time derivatives, the $\mathbf{r}_{B_c/B}$ and $\mathbf{r}_{S/B}$ vectors are fixed with respect to the $\mathcal{B}$ frame ($\mathbf{r}'_{B_c/B} = \mathbf{r}'_{S/B} = \mathbf{0}$), which means

$$m_{\text{sc}}\mathbf{c}' = m_{\text{S}}\mathbf{r}'_{S_c/B} = m_{\text{S}}\mathbf{r}'_{S_c/S} = m_{\text{S}}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \times \mathbf{r}_{S_c/S} \tag{5}$$
$$m_{\text{sc}}\mathbf{c}'' = m_{\text{S}}\left(\ddot{\theta}\hat{\boldsymbol{s}} \times \mathbf{r}_{S_c/S} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \times \mathbf{r}'_{S_c/S}\right) \tag{6}$$

where by definition:

$$\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{s}}, \qquad \boldsymbol{\omega}'_{\mathcal{S}/\mathcal{B}} = \ddot{\theta}\hat{\boldsymbol{s}} \tag{7}$$

because $\hat{\boldsymbol{s}}$ is fixed in the $\mathcal{B}$ frame. Finally, all these terms are combined to get

$$\begin{aligned} m_{\text{sc}}\ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} - m_{\text{sc}}[\tilde{\mathbf{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S}]\hat{\boldsymbol{s}}\ddot{\theta} = \mathbf{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c}' - \\ m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\mathbf{r}'_{S_c/S} \end{aligned} \tag{8}$$

In the equation above, the matrix cross product operator is used. For an arbitrary vector $\mathbf{a} = [a_1, a_2, a_3]^T$, the corresponding matrix cross product operator is written as $[\tilde{\mathbf{a}}]$ and is given by

$$[\tilde{\mathbf{a}}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{9}$$

**Rotational Equations of Motion**

For the rotational equation of motion, the entire spacecraft is considered. This equation of motion defines three degrees of freedom of the system. The rotational differential equation given about point $B$, which is not the system's center of mass, is given by

$$\dot{\mathbf{H}}_{\text{sc},B} = \mathbf{L}_B + m_{\text{sc}}\ddot{\mathbf{r}}_{B/N} \times \mathbf{c} \tag{10}$$

where $\mathbf{H}_{\text{sc},B}$ is the angular momentum of the spacecraft (sc) about point $B$ and $\mathbf{L}_B$ is the torque about point $B$. The angular momentum is

$$\mathbf{H}_{\text{sc},B} = \mathbf{H}_{\text{hub},B} + \mathbf{H}_{\text{S},B} = [I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + m_{\text{S}}\mathbf{r}_{S_c/B} \times \dot{\mathbf{r}}_{S_c/B} \tag{11}$$

where $\mathbf{H}_{\text{hub},B}$ is the angular momentum of the hub and $\mathbf{H}_{\text{S},B}$ is the angular momentum of the spinner, both about point $B$. The terms multiplied by $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ are grouped together to simplify the expression above. To express the inertial time derivative using the $\mathcal{B}$ frame time derivative, the equality $\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and $\dot{\mathbf{r}}_{S_c/B} = \mathbf{r}'_{S_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_c/B}$ is used, which yields

$$\mathbf{H}_{\text{sc},B} = [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + m_{\text{S}}\mathbf{r}_{S_c/B} \times \mathbf{r}'_{S_c/B} \tag{12}$$

where $[I_{\text{sc},B}] = [I_{\text{hub},B}] + [I_{\text{S},S_c}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}_{S_c/B}]$ is the spacecraft's total inertia about point $B$. The inertial time derivative of the total angular momentum is expressed as

$$\dot{\mathbf{H}}_{\text{sc},B} = [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\ddot{\theta}\hat{\boldsymbol{s}} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \times [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + \tag{13}$$

$$m_{\text{S}}\mathbf{r}_{S_c/B} \times \mathbf{r}''_{S_c/B} + m_{\text{S}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\mathbf{r}_{S_c/B} \times \mathbf{r}'_{S_c/B}) \tag{14}$$

Before writing the final equation, some terms are defined for compactness. To take the body-frame time derivative of the total spacecraft inertia, the inertia transport theorem needs to be used. The time derivative of the inertia tensor $[I]$ with respect to the $\mathcal{A}$ frame can be written using the time derivative with respect to the $\mathcal{B}$ frame as

$$\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}[I] = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}[I] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}}][I] - [I][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{A}}] \tag{15}$$

With this result, the body-frame time derivative of the total spacecraft inertia is

$$[I'_{\text{sc},B}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}][I_{\text{S},S_c}] - [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}] - m_{\text{S}}[\tilde{\mathbf{r}}'_{S_c/B}][\tilde{\mathbf{r}}_{S_c/B}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}'_{S_c/B}] \tag{16}$$

Since the $\mathbf{r}''_{S_c/B}$ contains second-order terms, it must be simplified

$$\mathbf{r}''_{S_c/B} = \mathbf{r}''_{S_c/S} = \ddot{\theta}\hat{\boldsymbol{s}} \times \mathbf{r}_{S_c/S} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \times \mathbf{r}'_{S_c/S} \tag{17}$$

Combining these results into the rotational equation of motion yields the final expression for the rotational equation of motion

$$\begin{aligned} m_{\text{sc}}[\tilde{\mathbf{c}}]\ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \left([I_{\text{S},S_c}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}_{S_c/S}]\right)\hat{\boldsymbol{s}}\ddot{\theta} = \\ = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}][I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} - \\ m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_c/B}]\mathbf{r}'_{S_c/B} - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\mathbf{r}'_{S_c/S} \end{aligned} \tag{18}$$

### Spinning Body Equations of Motion

For the final equation of motion, only the rotating rigid body is considered. This solves the final degree of freedom of the system. The general formulation of the equation of motion of the spinning body is

$$\dot{\mathbf{H}}_{\text{S},S} = \mathbf{L}_S - m_{\text{S}}\mathbf{r}_{S_c/S} \times \ddot{\mathbf{r}}_{S/N} \tag{19}$$

The angular momentum of the spinner about point $S$ is

$$\mathbf{H}_{\text{S},S} = [I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{20}$$

where $[I_{S,S}]$ is defined using the parallel axis theorem as $[I_{S,S}] = [I_{S,S_c}] - m_S[\tilde{\mathbf{r}}_{S_c/S}][\tilde{\mathbf{r}}_{S_c/S}]$. The inertial time derivative of the angular momentum is given by

$$\dot{\mathbf{H}}_{S,S} = [I_{S,S}]\dot{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \times [I_{S,S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{21}$$

As for the $\ddot{\mathbf{r}}_{S/N}$ term, we can separate it into two terms

$$\ddot{\mathbf{r}}_{S/N} = \ddot{\mathbf{r}}_{S/B} + \ddot{\mathbf{r}}_{B/N} \tag{22}$$

To compute $\ddot{\mathbf{r}}_{S/B}$, the fact that $\mathbf{r}_{S/B}$ is constant in the $\mathcal{B}$ frame is used to yield

$$\dot{\mathbf{r}}_{S/B} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S/B}, \qquad \ddot{\mathbf{r}}_{S/B} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\mathbf{r}}_{S/B} \tag{23}$$

The term $\dot{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}$ can be separated into three distinct terms

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \ddot{\theta}\hat{\boldsymbol{s}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \tag{24}$$

Before these results are combined, the dot product with the spin axis $\hat{s}$ is applied to all terms. This removes the contributions in other directions apart from the rotating axis. Implicitly, structural torques are applied to the other directions to keep the single-axis rotation constraint in place. This results in the following equation of motion:

$$\begin{aligned}\hat{\boldsymbol{s}}^T[I_{S,S}]\hat{\boldsymbol{s}}\ddot{\theta} = \mathbf{u}_S - m_S\hat{\boldsymbol{s}}^T[\tilde{\mathbf{r}}_{S_c/S}]\ddot{\mathbf{r}}_{B/N} - \hat{\boldsymbol{s}}^T\left([I_{S,S}] - m_S[\tilde{\mathbf{r}}_{S_c/S}][\tilde{\mathbf{r}}_{S/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \\ \hat{\boldsymbol{s}}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}][I_{S,S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} - \hat{\boldsymbol{s}}^T[I_{S,S}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} - m_S\hat{\boldsymbol{s}}^T[\tilde{\mathbf{r}}_{S_c/S}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\dot{\mathbf{r}}_{S/B}\end{aligned} \tag{25}$$

### Back-Substitution Formulation

The modular software implementation of these equations of motion requires that they are written in a particular way. The full explanation of this approach is given in Ref. 6. The spinning body equation of motion can be written in the form

$$\boldsymbol{m}_\theta\ddot{\theta} = \boldsymbol{a}_\theta^* \cdot \ddot{\mathbf{r}}_{B/N} + \boldsymbol{b}_\theta^* \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_\theta^* \tag{26}$$

where the following terms are introduced

$$\boldsymbol{a}_\theta^* = m_S[\tilde{\mathbf{r}}_{S_c/S}]\hat{\boldsymbol{s}} \tag{27}$$

$$\boldsymbol{b}_\theta^* = -\left([I_{S,S}] - m_S[\tilde{\mathbf{r}}_{S/B}][\tilde{\mathbf{r}}_{S_c/S}]\right)\hat{\boldsymbol{s}} \tag{28}$$

$$\boldsymbol{c}_\theta^* = \mathbf{u}_S - \hat{\boldsymbol{s}}^T\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}][I_{S,S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [I_{S,S}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + m_S[\tilde{\mathbf{r}}_{S_c/S}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\dot{\mathbf{r}}_{S/B}\right) \tag{29}$$

along with the mass-like term $\boldsymbol{m}_\theta = \hat{\boldsymbol{s}}^T[I_{S,S}]\hat{\boldsymbol{s}}$. Using these terms, the spinning body equation of motion can be written in its compact form as

$$\ddot{\theta} = \boldsymbol{a}_\theta \cdot \ddot{\mathbf{r}}_{B/N} + \boldsymbol{b}_\theta \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_\theta \tag{30}$$

where the new variables are defined as

$$\boldsymbol{a}_\theta^* = \frac{\boldsymbol{a}_\theta^*}{\boldsymbol{m}_\theta}, \qquad \boldsymbol{b}_\theta^* = \frac{\boldsymbol{b}_\theta^*}{\boldsymbol{m}_\theta}, \qquad \boldsymbol{c}_\theta^* = \frac{\boldsymbol{c}_\theta^*}{\boldsymbol{m}_\theta} \tag{31}$$

This result can be backsubstituted into equations (8) and (18), which yields

$$\begin{bmatrix}[A] & [B] \\ [C] & [D]\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}}\end{bmatrix} \tag{32}$$

using the following matrices

$$[A] = m_{\text{SC}}[I_{3\times3}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S}]\hat{\boldsymbol{s}}\boldsymbol{a}_\theta^T \tag{33}$$

$$[B] = -m_{\text{sc}}[\tilde{\mathbf{c}}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S}]\hat{\boldsymbol{s}}\boldsymbol{b}_\theta^T \tag{34}$$

$$[C] = m_{\text{sc}}[\tilde{\mathbf{c}}] + \left([I_{\text{S},S_c}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}_{S_c/S}]\right)\hat{\boldsymbol{s}}a_\theta^T \tag{35}$$

$$[D] = [I_{\text{sc},B}] + \left([I_{\text{S},S_c}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}_{S_c/S}]\right)\hat{\boldsymbol{s}}b_\theta^T \tag{36}$$

and vectors

$$\mathbf{v}_{\text{trans}} = \mathbf{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\mathbf{r}'_{S_c/S} + m_{\text{S}}\boldsymbol{c}_\theta[\tilde{\mathbf{r}}_{S_c/S}]\hat{\boldsymbol{s}} \tag{37}$$

$$\begin{split}\mathbf{v}_{\text{rot}} = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}][I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} - \\ m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_c/B}]\mathbf{r}'_{S_c/B} - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\mathbf{r}'_{S_c/S} - \boldsymbol{c}_\theta\left([I_{\text{S},S_c}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/B}][\tilde{\mathbf{r}}_{S_c/S}]\right)\hat{\boldsymbol{s}}\end{split} \tag{38}$$

### Validation

Validation is a crucial step in implementing the equations of motion. It is impossible to guarantee that the equations are correct and implemented appropriately without validating the approach used. The validation method verifies that some conservation laws are being respected by checking whether some physical quantities remain constant throughout the simulation. While this alone does not guarantee that the equations are correct, it gives high confidence that they have been correctly derived and implemented.

The quantities being verified are the orbital energy, the orbital angular momentum, the rotational energy, and the rotational angular momentum. In the presence of gravity, a conservative force, energy should be constant. Moreover, since gravity is a radial force, the orbital angular momentum is also constant throughout the simulation. The rotational quantities should also remain constant without torques and non-conservative forces. The complete derivation and explanation of why these quantities must be conserved are given in Ref. 6.

The validation results are given in Figures 4, 5, 6 and 7. It should be noted that while the plots do not immediately look constant, the scale on the vertical axis is on the order of $10^{-15}$ to $10^{-14}$. This is very close to machine precision, which means numerical errors slightly corrupt the data. Moreover, the random walk in these plots is very common in fixed-step integrators like the fourth-order Runge-Kutta used in these simulations.

## DUAL-AXIS ROTATING RIGID BODY DYNAMICS

This section shows the derivation and validation of the equations of motion of a dual-axis rotating rigid body attached to a rigid hub. The two-degree-of-freedom component can be described in one of two ways: first, as a chain of two rigid bodies connected to each other by rotary joints, each rotating about a particular spin axis; second, as a single rigid body connected to the hub through a universal joint, which can have two spin axes.

This description can represent various common spacecraft components. Examples include dual-hinge solar arrays for deployment or second-order flexing analysis, control moment gyroscopes as

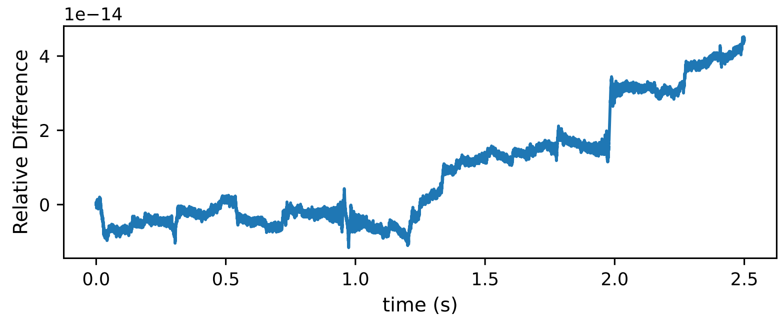


**Figure 4. Orbital energy using a single-axis rotating rigid body.**

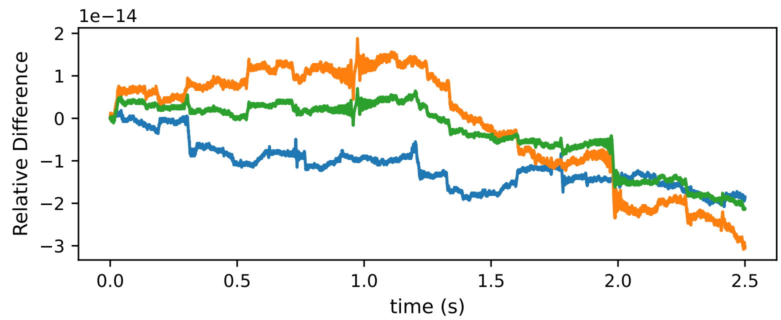


**Figure 5. Orbital angular momentum using a single-axis rotating rigid body.**

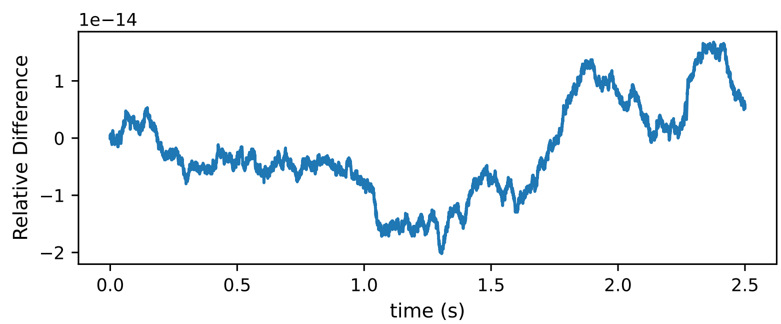


**Figure 6. Rotational energy using a single-axis rotating rigid body.**

attitude control devices, and two-axis gimbaled high-gain antennas. All these components can be defined through a general description, where they are specified by their masses, inertia matrices, the location of the centers of mass, and spin axes.

### Problem Statement

The problem statement for the dual-axis rotating rigid body is given in Figure 8. The inertial frame is represented by $\mathcal{N}$ with origin at point $N$. The spacecraft is composed of two rigid bodies connected to each other and to a rigid hub through two axes of rotation. The hub has a body-fixed

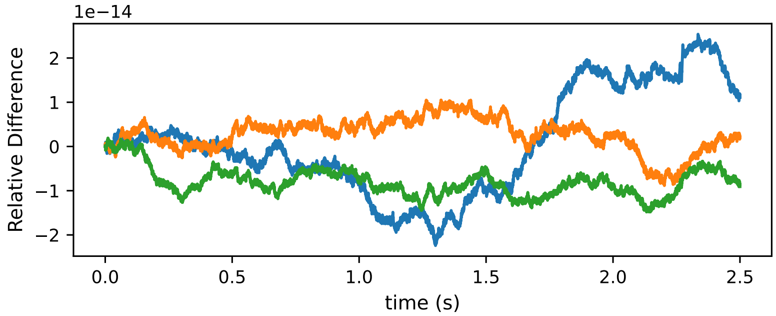


**Figure 7. Rotational angular momentum using a single-axis rotating rigid body.**

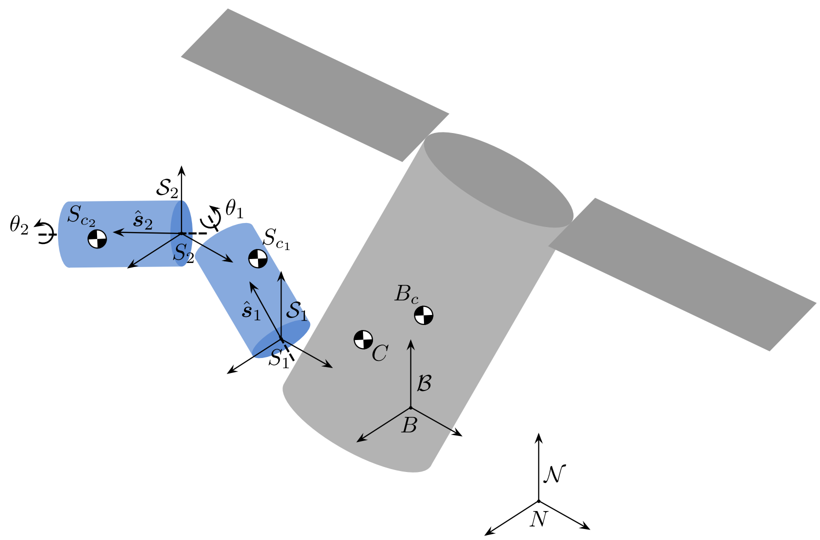

**Figure 8. Problem statement for the two-degree-of-freedom spinning rigid body.**

frame $\mathcal{B}$ with origin $B$, and its center of mass is located at point $B_c$. The mass of the hub is $m_{\text{hub}}$, and its inertia matrix about point $B$ is $[I_{\text{hub},B}]$. The lower rotating rigid body has the $\mathcal{S}_1$ frame attached to it with its origin at point $S_1$ and its center of mass located at point $S_{c_1}$. The mass of the lower body is $m_{\text{S}_1}$, and its inertia matrix about its center of mass is $[I_{\text{S}_1,S_{c_1}}]$. The upper rotating rigid body has the $\mathcal{S}_2$ frame attached to it with its origin at point $S_2$ and its center of mass located at point $S_{c_2}$. The mass of the upper body is $m_{\text{S}_2}$, and its inertia matrix about its center of mass is $[I_{\text{S}_2,S_{c_2}}]$. The center of mass of the spinning system is located at point $S_c$, and its mass is $m_{\text{S}}$. The combined center of mass of the spacecraft is located at point $C$. The first spin axis $\hat{\boldsymbol{s}}_1$ is constant, as seen by the $\mathcal{B}$ frame, and passes through the point $S_1$. The angle about this rotation axis is $\theta_1$, and its angle rate is $\dot{\theta}_1$. The second spin axis $\hat{\boldsymbol{s}}_2$ is constant, as seen by the $\mathcal{S}_1$ frame, and passes through the point $S_2$. The angle about this rotation axis is $\theta_2$, and its angle rate is $\dot{\theta}_2$.

The two-body description is used to describe the two-axis rotating rigid body system as generally as possible. However, the resulting equations of motion still apply to a single rotating body attached by a universal joint. To do this, the user can set the mass and inertia matrix of the lower body to zero, which does not impart any singularity in the equations.

The dual-axis rotating rigid body attached to the hub has eight degrees of freedom shown in Table 2: three for the system's position, three for the system's attitude, and one for the angle about each rotation axis. Like the one-degree-of-freedom case, the motion equations are developed to describe all eight degrees of freedom. Therefore, beyond the translational and rotational equations of motion, the system needs two spinner equations to describe each angle.

| State Variables | Degrees of Freedom | Equations of motion |
|---|---|---|
| $\mathbf{r}_{B/N}, \dot{\mathbf{r}}_{B/N}$ | 3 | Translational |
| $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}, \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ | 3 | Rotational |
| $\theta_1, \dot{\theta}_1$ | 1 | First Spinner Rotational |
| $\theta_2, \dot{\theta}_2$ | 1 | Second Spinner Rotational |

**Table 2. State variables for the single-axis rotating rigid body spacecraft.**

### Translational Equations of Motion

For the translational equations of motion, the entire spacecraft is considered. This describes three degrees of freedom. Using the Super Particle Theorem:

$$m_{\text{sc}}\ddot{\mathbf{r}}_{C/N} = m_{\text{sc}}\ddot{\mathbf{r}}_{B/N} + m_{\text{sc}}\ddot{\mathbf{c}} = \mathbf{F} \tag{39}$$

where $\mathbf{c} \equiv \mathbf{r}_{C/B}$. Using the definition of the center of mass of the system:

$$m_{\text{sc}}\mathbf{c} = m_{\text{hub}}\mathbf{r}_{B_c/B} + m_{\text{S}_1}\mathbf{r}_{S_{c_1}/B} + m_{\text{S}_2}\mathbf{r}_{S_{c_2}/B} \tag{40}$$

Using the transport theorem, the inertial time derivatives can be expressed using body-frame derivatives as

$$\dot{\mathbf{c}} = \mathbf{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c} \tag{41}$$

$$\ddot{\mathbf{c}} = \mathbf{c}'' + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{c}\right) \tag{42}$$

The first-order body-frame derivatives for the three terms that are part of $\mathbf{c}$ are given by

$$\mathbf{r}'_{B_c/B} = \mathbf{0} \tag{43}$$

$$\mathbf{r}'_{S_{c_1}/B} = \mathbf{r}'_{S_{c_1}/S_1} = \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \mathbf{r}_{S_{c_1}/S_1} \tag{44}$$

$$\mathbf{r}'_{S_{c_2}/B} = \mathbf{r}'_{S_{c_2}/S_2} + \mathbf{r}'_{S_2/S_1} = \boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} \times \mathbf{r}_{S_{c_2}/S_2} + \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \mathbf{r}_{S_2/S_1} \tag{45}$$

where, by definition, $\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} = \dot{\theta}_1\hat{\boldsymbol{s}}_1$ and $\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} = \boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} + \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} = \dot{\theta}_2\hat{\boldsymbol{s}}_2 + \dot{\theta}_1\hat{\boldsymbol{s}}_1$. The second-order body-frame derivatives are given by

$$\mathbf{r}''_{B_c/B} = \mathbf{0} \tag{46}$$

$$\mathbf{r}''_{S_{c_1}/B} = \ddot{\theta}_1\hat{\boldsymbol{s}}_1 \times \mathbf{r}_{S_{c_1}/S_1} + \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \mathbf{r}'_{S_{c_1}/S_1} \tag{47}$$

$$\begin{aligned}\mathbf{r}''_{S_{c_2}/B} = \ddot{\theta}_2\hat{\boldsymbol{s}}_2 \times \mathbf{r}_{S_{c_2}/S_2} + \ddot{\theta}_1\hat{\boldsymbol{s}}_1 \times \mathbf{r}_{S_{c_2}/S_1} + \left(\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}\right) \times \mathbf{r}_{S_{c_2}/S_2} + \\ \boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} \times \mathbf{r}'_{S_{c_2}/S_2} + \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \mathbf{r}'_{S_{c_2}/S_1}\end{aligned} \tag{48}$$

where $\boldsymbol{\omega}'_{\mathcal{S}_1/\mathcal{B}} = \ddot{\theta}_1 \hat{\boldsymbol{s}}_1$ and $\boldsymbol{\omega}'_{\mathcal{S}_2/\mathcal{B}} = \ddot{\theta}_1 \hat{\boldsymbol{s}}_1 + \ddot{\theta}_2 \hat{\boldsymbol{s}}_2 + \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} \times \boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}$ because $\hat{\boldsymbol{s}}_1$ is fixed in the $\mathcal{B}$ frame and $\hat{\boldsymbol{s}}_2$ is fixed in the $\mathcal{S}_1$ frame. With these results, the expressions for $m_{\text{sc}}\mathbf{c}'$ and $m_{\text{sc}}\mathbf{c}''$ are

$$m_{\text{sc}}\mathbf{c}' = m_{\text{S}_1}\mathbf{r}'_{S_{c_1}/B} + m_{\text{S}_2}\mathbf{r}'_{S_{c_2}/B} = m_{\text{S}}\mathbf{r}'_{S_c/B} \tag{49}$$

$$\begin{aligned} m_{\text{sc}}\mathbf{c}'' = m_{\text{S}_1}\mathbf{r}''_{S_{c_1}/B} + m_{\text{S}_2}\mathbf{r}''_{S_{c_2}/B} = -m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}]\hat{\boldsymbol{s}}_1\ddot{\theta}_1 - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\ddot{\theta}_2 + \\ m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\mathbf{r}'_{S_c/B} + m_{\text{S}_2}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} - [\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}\right) \end{aligned} \tag{50}$$

The center of mass of the spinning bodies system about point $S_1$ is defined as $m_{\text{S}}\mathbf{r}_{S_c/S_1} = m_{\text{S}_1}\mathbf{r}_{S_{c_1}/S_1} + m_{\text{S}_2}\mathbf{r}_{S_{c_2}/S1}$. Finally, combining similar terms together yields

$$\begin{aligned} m_{\text{sc}}\ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} - m_{\text{sc}}[\tilde{\mathbf{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}]\hat{\boldsymbol{s}}_1\ddot{\theta}_1 - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\ddot{\theta}_2 = \\ \mathbf{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\mathbf{r}'_{S_c/B} - \\ m_{\text{S}_2}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} - [\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}\right) \end{aligned} \tag{51}$$

**Rotational Equations of Motion**

For the rotational equation of motion, the entire spacecraft is considered. Three degrees of freedom of the system are described by this equation of motion. The rotational differential equation given about point $B$, which is not the system's center of mass, is given by

$$\dot{\mathbf{H}}_{\text{sc},B} = \mathbf{L}_B + m_{\text{sc}}\ddot{\mathbf{r}}_{B/N} \times \mathbf{c} \tag{52}$$

The angular momentum about point $B$ is

$$\begin{aligned} \mathbf{H}_{\text{sc},B} = \mathbf{H}_{\text{hub},B} + \mathbf{H}_{\text{S}_1,B} + \mathbf{H}_{\text{S}_2,B} = \\ = [I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{N}} + m_{\text{S}_1}\mathbf{r}_{S_{c_1}/B} \times \dot{\mathbf{r}}_{S_{c_1}/B} + \\ [I_{\text{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} + m_{\text{S}_2}\mathbf{r}_{S_{c_2}/B} \times \dot{\mathbf{r}}_{S_{c_2}/B} \end{aligned} \tag{53}$$

As before, it is useful to express the inertial time derivative using the $\mathcal{B}$ frame derivative and the transport theorem by noting that $\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and $\dot{\mathbf{r}}_{S_{c_1}/B} = \mathbf{r}'_{S_{c_1}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_{c_1}/B}$. An equivalent development can be done for the second spinning body. Grouping the terms multiplied by each angular velocity yields

$$\begin{aligned} \mathbf{H}_{\text{sc},B} = [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + [I_{\text{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + \\ m_{\text{S}_1}\mathbf{r}_{S_{c_1}/B} \times \mathbf{r}'_{S_{c_1}/B} + m_{\text{S}_2}\mathbf{r}_{S_{c_2}/B} \times \mathbf{r}'_{S_{c_2}/B} \end{aligned} \tag{54}$$

where $[I_{\text{sc},B}] = [I_{\text{hub},B}] + [I_{\text{S}_1,S_{c_1}}] + [I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/B}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/B}]$ is the spacecraft's total inertia about point $B$. To take the inertial time derivative of the total angular momentum, the transport theorem is used to take the body-frame time derivatives instead, which yields

$$\dot{\mathbf{H}}_{\text{sc},B} = \mathbf{H}'_{\text{sc},B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{H}_{\text{sc},B} \tag{55}$$

The body-frame derivative of the angular momentum is

$$\begin{aligned} \mathbf{H}'_{\text{sc},B} = [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I'_{\text{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + [I_{\text{S}_1,S_{c_1}}]\boldsymbol{\omega}'_{\mathcal{S}_1/\mathcal{B}} + \\ [I'_{\text{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + [I_{\text{S}_2,S_{c_2}}]\boldsymbol{\omega}'_{\mathcal{S}_2/\mathcal{B}} + m_{\text{S}_1}\mathbf{r}_{S_{c_1}/B} \times \mathbf{r}''_{S_{c_1}/B} + m_{\text{S}_2}\mathbf{r}_{S_{c_2}/B} \times \mathbf{r}''_{S_{c_2}/B} \end{aligned} \tag{56}$$

where the derivative product rule is applied. To simplify the expression above, the body-frame derivatives of the inertia matrices are defined using the inertia transport theorem

$$[I'_{\mathrm{S}_1,S_{c_1}}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}][I_{\mathrm{S}_1,S_{c_1}}] - [I_{\mathrm{S}_1,S_{c_1}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] \tag{57}$$

$$[I'_{\mathrm{S}_2,S_{c_2}}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{B}}][I_{\mathrm{S}_2,S_{c_2}}] - [I_{\mathrm{S}_2,S_{c_2}}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{B}}] \tag{58}$$

$$\begin{aligned}[I'_{\mathrm{sc},B}] = [I'_{\mathrm{S}_1,S_{c_1}}] + [I'_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_1}\Big([\tilde{\mathbf{r}}'_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/B}] + [\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}'_{S_{c_1}/B}]\Big) - \\ m_{\mathrm{S}_2}\Big([\tilde{\mathbf{r}}'_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/B}] + [\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}'_{S_{c_2}/B}]\Big)\end{aligned} \tag{59}$$

Combining these results with the definitions derived in the translational equation of motion sections yields the following rotational equation of motion

$$\begin{aligned} m_{\mathrm{sc}}[\tilde{\mathbf{c}}]\ddot{\mathbf{r}}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \\ \Big([I_{\mathrm{S}_1,S_{c_1}}] + [I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\Big)\hat{\boldsymbol{s}}_1\ddot{\theta}_1 + \\ \Big([I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\Big)\hat{\boldsymbol{s}}_2\ddot{\theta}_2 = \mathbf{L}_B - \big([I'_{\mathrm{sc},B}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{sc},B}]\big)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \\ -\Big([I'_{\mathrm{S}_1,S_{c_1}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{S}_1,S_{c_1}}]\Big)\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} - \Big([I'_{\mathrm{S}_2,S_{c_2}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{S}_2,S_{c_2}}]\Big)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} - \\ -\Big([I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\Big)[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} - \\ m_{\mathrm{S}_1}\Big([\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_1}/B}]\Big)\mathbf{r}'_{S_{c_1}/B} - \\ -m_{\mathrm{S}_2}\Big([\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_2}/B}]\Big)\mathbf{r}'_{S_{c_2}/B} - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2}\end{aligned} \tag{60}$$

**First Spinning Body Equations of Motion**

The first spinning body equation of motion describes the motion of the spinning body system, defining another degree of freedom of the spacecraft. The formulation of the equation of motion for the spinning body system is

$$\dot{\mathbf{H}}_{\mathrm{S},S_1} = \mathbf{L}_{S_1} - m_{\mathrm{S}}\mathbf{r}_{S_c/S_1} \times \ddot{\mathbf{r}}_{S_1/N} \tag{61}$$

The angular momentum of the spinning system is

$$\begin{aligned}\mathbf{H}_{\mathrm{S},S_1} = \mathbf{H}_{\mathrm{S}_1,S_1} + \mathbf{H}_{\mathrm{S}_2,S_1} = \\ = [I_{\mathrm{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{N}} + m_{\mathrm{S}_1}\mathbf{r}_{S_{c_1}/S_1} \times \dot{\mathbf{r}}_{S_{c_1}/S_1} + [I_{\mathrm{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} + m_{\mathrm{S}_2}\mathbf{r}_{S_{c_2}/S_1} \times \dot{\mathbf{r}}_{S_{c_2}/S_1}\end{aligned} \tag{62}$$

The expression above can be simplified by applying the transport theorem to the $\dot{\mathbf{r}}$ terms and grouping the $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ terms as follows

$$\begin{aligned}\mathbf{H}_{\mathrm{S},S_1} = [I_{\mathrm{S},S_1}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + [I_{\mathrm{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + \\ m_{\mathrm{S}_1}\mathbf{r}_{S_{c_1}/S_1} \times \mathbf{r}'_{S_{c_1}/S_1} + m_{\mathrm{S}_2}\mathbf{r}_{S_{c_2}/S_1} \times \mathbf{r}'_{S_{c_2}/S_1}\end{aligned} \tag{63}$$

where three new inertia matrices are defined

$$[I_{\mathrm{S},S_1}] = [I_{\mathrm{S}_1,S_1}] + [I_{\mathrm{S}_2,S_1}] \tag{64}$$

$$[I_{\mathrm{S}_1,S_1}] = [I_{\mathrm{S}_1,S_{c_1}}] - m_{\mathrm{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/S_1}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] \tag{65}$$

$$[I_{\mathrm{S}_2,S_1}] = [I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}] \tag{66}$$

Using the transport theorem to take the derivatives in the $\mathcal{B}$ frame, the inertial time derivative of the angular momentum is given by

$$\dot{\mathbf{H}}_{\mathrm{S},S_1} = \mathbf{H}'_{\mathrm{S},S_1} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{H}_{\mathrm{S},S_1} \tag{67}$$

The body-frame derivative of the angular momentum is

$$\begin{aligned} \mathbf{H}'_{\mathrm{S},S_1} = [I'_{\mathrm{S},S_1}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{S},S_1}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I'_{\mathrm{S}_1,S_{c_1}}]\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + [I_{\mathrm{S}_1,S_{c_1}}]\boldsymbol{\omega}'_{\mathcal{S}_1/\mathcal{B}} + \\ [I'_{\mathrm{S}_2,S_{c_2}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + [I_{\mathrm{S}_2,S_{c_2}}]\boldsymbol{\omega}'_{\mathcal{S}_2/\mathcal{B}} + +m_{\mathrm{S}_1}\mathbf{r}_{S_{c_1}/S_1} \times \mathbf{r}''_{S_{c_1}/S_1} + m_{\mathrm{S}_2}\mathbf{r}_{S_{c_2}/S_1} \times \mathbf{r}''_{S_{c_2}/S_1} \end{aligned} \tag{68}$$

The body-frame derivatives of the inertia matrices are

$$[I'_{\mathrm{S},S_1}] = [I'_{\mathrm{S}_1,S_1}] + [I'_{\mathrm{S}_2,S_1}] \tag{69}$$

$$[I'_{\mathrm{S}_1,S_1}] = [I'_{\mathrm{S}_1,S_{c_1}}] - m_{\mathrm{S}_1}\left([\tilde{\mathbf{r}}'_{S_{c_1}/S_1}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] + [\tilde{\mathbf{r}}_{S_{c_1}/S_1}][\tilde{\mathbf{r}}'_{S_{c_1}/S_1}]\right) \tag{70}$$

$$[I'_{\mathrm{S}_2,S_1}] = [I'_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}\left([\tilde{\mathbf{r}}'_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}] + [\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}'_{S_{c_2}/S_1}]\right) \tag{71}$$

As for the $\ddot{\mathbf{r}}_{S_1/N}$ term, it can be separated into two terms

$$\ddot{\mathbf{r}}_{S_1/N} = \ddot{\mathbf{r}}_{S_1/B} + \ddot{\mathbf{r}}_{B/N} \tag{72}$$

To compute $\ddot{\mathbf{r}}_{S_1/B}$, it should be noted that $\mathbf{r}_{S_1/B}$ is constant in the $\mathcal{B}$ frame, which yields

$$\dot{\mathbf{r}}_{S_1/B} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_1/B}, \qquad \ddot{\mathbf{r}}_{S_1/B} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_1/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\mathbf{r}}_{S_1/B} \tag{73}$$

Here, all terms are dotted with the spin axis $\hat{\boldsymbol{s}}_1$ to ignore the dynamics in any other direction, where structural torques keep the constraints in place. This results in the first spinning body equation of motion

$$\begin{aligned} \hat{\boldsymbol{s}}_1^T[I_{\mathrm{S},S_1}]\hat{\boldsymbol{s}}_1\ddot{\theta}_1 + \hat{\boldsymbol{s}}_1^T\left([I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)\hat{\boldsymbol{s}}_2\ddot{\theta}_2 = \boldsymbol{u}_{S_1} - \\ m_{\mathrm{S}}\hat{\boldsymbol{s}}_1^T[\tilde{\mathbf{r}}_{S_c/S_1}]\ddot{\mathbf{r}}_{B/N} - \hat{\boldsymbol{s}}_1^T\left([I_{\mathrm{S},S_1}] - m_{\mathrm{S}}[\tilde{\mathbf{r}}_{S_c/S_1}][\tilde{\mathbf{r}}_{S_1/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \\ \hat{\boldsymbol{s}}_1^T\left([I'_{\mathrm{S},S_1}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{S},S_1}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \hat{\boldsymbol{s}}_1^T\left([I'_{\mathrm{S}_1,S_{c_1}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{S}_1,S_{c_1}}]\right)\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} - \\ \hat{\boldsymbol{s}}_1^T\left([I'_{\mathrm{S}_2,S_{c_2}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{S}_2,S_{c_2}}]\right)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} - \\ \hat{\boldsymbol{s}}_1^T\left([I_{\mathrm{S}_2,S_{c_2}}] - m_{\mathrm{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} - \\ m_{\mathrm{S}_1}\hat{\boldsymbol{s}}_1^T\left([\tilde{\mathbf{r}}_{S_{c_1}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}]\right)\mathbf{r}'_{S_{c_1}/S_1} - \\ m_{\mathrm{S}_2}\hat{\boldsymbol{s}}_1^T\left([\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\right)\mathbf{r}'_{S_{c_2}/S_1} - \\ m_{\mathrm{S}_2}\hat{\boldsymbol{s}}_1^T[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} - m_{\mathrm{S}}\hat{\boldsymbol{s}}_1^T[\tilde{\mathbf{r}}_{S_c/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\dot{\mathbf{r}}_{S_1/B} \end{aligned} \tag{74}$$

**Second Spinning Body Equations of Motion**

For the final equation of motion, only the top spinner is considered, describing the last degree of freedom of the system. The formulation for the equation of motion for the second spinner is

$$\dot{\mathbf{H}}_{\mathrm{S}_2,S_2} = \mathbf{L}_{S_2} - m_{\mathrm{S}_2}\mathbf{r}_{S_{c_2}/S_2} \times \ddot{\mathbf{r}}_{S_2/N} \tag{75}$$

The angular momentum of the top spinner about point $S_2$ is

$$\mathbf{H}_{\text{S}_2,S_2} = [I_{\text{S}_2,S_2}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} = [I_{\text{S}_2,S_2}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S}_2,S_2}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} \tag{76}$$

where $[I_{\text{S}_2,S_2}]$ is defined as $[I_{\text{S}_2,S_2}] = [I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]$ using the parallel axis theorem. The inertial time derivative of the angular momentum is given by

$$\dot{\mathbf{H}}_{S_2,S_2} = \mathbf{H}'_{S_2,S_2} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{H}_{S_2,S_2} \tag{77}$$

The body-frame time derivative of $\mathbf{H}_{S_2,S_2}$ is

$$\mathbf{H}'_{S_2,S_2} = [I'_{\text{S}_2,S_2}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S}_2,S_2}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I'_{\text{S}_2,S_2}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + [I_{\text{S}_2,S_2}]\boldsymbol{\omega}'_{\mathcal{S}_2/\mathcal{B}} \tag{78}$$

As for the $\ddot{\mathbf{r}}_{S_2/N}$ term, it can be separated into three terms

$$\ddot{\mathbf{r}}_{S_2/N} = \ddot{\mathbf{r}}_{S_2/S_1} + \ddot{\mathbf{r}}_{S_1/B} + \ddot{\mathbf{r}}_{B/N} \tag{79}$$

where $\ddot{\mathbf{r}}_{S_2/S_1}$ is equal to

$$\dot{\mathbf{r}}_{S_2/S_1} = \mathbf{r}'_{S_2/S_1} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_2/S_1} \tag{80}$$

$$\ddot{\mathbf{r}}_{S_2/S_1} = \mathbf{r}''_{S_2/S_1} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}_{S_2/S_1} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r}'_{S_2/S_1} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\mathbf{r}}_{S_2/S_1} \tag{81}$$

These results can be combined into the spinning body equation of motion by dotting each term with $\hat{\boldsymbol{s}}_2$:

$$\begin{gathered}\hat{\boldsymbol{s}}_2^T \left([I_{\text{S}_2,S_2}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\mathbf{r}}_{S_2/S_1}]\right)\hat{\boldsymbol{s}}_1\ddot{\theta}_1 + \hat{\boldsymbol{s}}_2^T[I_{\text{S}_2,S_2}]\hat{\boldsymbol{s}}_2\ddot{\theta}_2 = \boldsymbol{u}_{S_2} - \\ m_{\text{S}_2}\hat{\boldsymbol{s}}_2^T[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\ddot{\mathbf{r}}_{B/N} - \hat{\boldsymbol{s}}_2^T\left([I_{\text{S}_2,S_2}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\mathbf{r}}_{S_2/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \\ \hat{\boldsymbol{s}}_2^T\left([I'_{\text{S}_2,S_2}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_2,S_2}]\right)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} - \hat{\boldsymbol{s}}_2^T[I_{\text{S}_2,S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} - \\ m_{\text{S}_2}\hat{\boldsymbol{s}}_2^T[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{N}}]\mathbf{r}'_{S_2/S_1} - m_{\text{S}_2}\hat{\boldsymbol{s}}_2^T[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(\dot{\mathbf{r}}_{S_2/S_1} + \dot{\mathbf{r}}_{S_1/B}\right)\end{gathered} \tag{82}$$

**Back-Substitution Formulation**

To get a compact formulation for both equations of motion of both rotating rigid bodies, they are expressed in matrix form as such

$$[\boldsymbol{M}_\theta]\ddot{\boldsymbol{\theta}} = [\boldsymbol{A}^*_\theta]\ddot{\mathbf{r}}_{B/N} + [B^*_\theta]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{C}^*_\theta] \tag{83}$$

where the matrices above are defined as

$$[\boldsymbol{M}_\theta] = \begin{bmatrix} \hat{\boldsymbol{s}}_1^T[I_{\text{S},S_1}]\hat{\boldsymbol{s}}_1 & \hat{\boldsymbol{s}}_1^T\left([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)\hat{\boldsymbol{s}}_2 \\ \hat{\boldsymbol{s}}_2^T\left([I_{\text{S}_2,S_2}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\mathbf{r}}_{S_2/S_1}]\right)\hat{\boldsymbol{s}}_1 & \hat{\boldsymbol{s}}_2^T[I_{\text{S}_2,S_2}]\hat{\boldsymbol{s}}_2 \end{bmatrix} \tag{84}$$

$$[\boldsymbol{A}_\theta^*] = \begin{bmatrix} -m_{\text{S}}\hat{\boldsymbol{s}}_1^T[\tilde{\mathbf{r}}_{S_c/S_1}] \\ -m_{\text{S}_2}\hat{\boldsymbol{s}}_2^T[\tilde{\mathbf{r}}_{S_{c_2}/S_2}] \end{bmatrix} \tag{85}$$

$$[\boldsymbol{B}_\theta^*] = \begin{bmatrix} -\hat{\boldsymbol{s}}_1^T\left([I_{\text{S},S_1}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}][\tilde{\mathbf{r}}_{S_1/B}]\right) \\ -\hat{\boldsymbol{s}}_2^T\left([I_{\text{S}_2,S_2}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\mathbf{r}}_{S_2/B}]\right) \end{bmatrix} \tag{86}$$

$$[\boldsymbol{C}_\theta^*] = \begin{bmatrix} -\hat{\boldsymbol{s}}_1^T\Big\{\left([I'_{\text{S},S_1}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_1}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \left([I'_{\text{S}_1,S_{c_1}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_1,S_{c_1}}]\right)\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} + \\ \left([I'_{\text{S}_2,S_{c_2}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_2,S_{c_2}}]\right)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} + \\ \left([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} + \\ m_{\text{S}_1}\left([\tilde{\mathbf{r}}_{S_{c_1}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}]\right)\mathbf{r}'_{S_{c_1}/S_1} + \\ m_{\text{S}_2}\left([\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\right)\mathbf{r}'_{S_{c_2}/S_1} + \\ m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} + m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\dot{\mathbf{r}}_{S_1/B}\Big\} \\ \\ -\hat{\boldsymbol{s}}_2^T\Big\{\left([I'_{\text{S}_2,S_2}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_2,S_2}]\right)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{N}} + [I_{\text{S}_2,S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} + \\ m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{N}}]\mathbf{r}'_{S_2/S_1} + m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(\dot{\mathbf{r}}_{S_2/S_1} + \dot{\mathbf{r}}_{S_1/B}\right)\Big\} \end{bmatrix} \tag{87}$$

The canonical form of equation (83) is given by

$$\ddot{\theta} = [\boldsymbol{A}_\theta]\ddot{\mathbf{r}}_{B/N} + [B_\theta]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{C}_\theta] \tag{88}$$

where the new matrices are defined as

$$[\boldsymbol{A}_\theta] = [\boldsymbol{M}_\theta]^{-1}[\boldsymbol{A}_\theta^*], \qquad [\boldsymbol{B}_\theta] = [\boldsymbol{M}_\theta]^{-1}[\boldsymbol{B}_\theta^*], \qquad [\boldsymbol{C}_\theta] = [\boldsymbol{M}_\theta]^{-1}[\boldsymbol{C}_\theta^*] \tag{89}$$

These results can be plugged into the back-substitution formulation as such

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}} \end{bmatrix} \tag{90}$$

using the following matrices

$$[A] = m_{\text{SC}}[I_{3\times 3}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}]\hat{\boldsymbol{s}}_1\boldsymbol{A}_{\theta_1} - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\boldsymbol{A}_{\theta_2} \tag{91}$$

$$[B] = -m_{\text{sc}}[\tilde{\mathbf{c}}] - m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}]\hat{\boldsymbol{s}}_1\boldsymbol{B}_{\theta_1} - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\boldsymbol{B}_{\theta_2} \tag{92}$$

$$\begin{aligned}[C] = m_{\text{sc}}[\tilde{\mathbf{c}}] + \Big([I_{\text{S}_1,S_{c_1}}] + [I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] - \\ m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\Big)\hat{\boldsymbol{s}}_1\boldsymbol{A}_{\theta_1} + \Big([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\Big)\hat{\boldsymbol{s}}_2\boldsymbol{A}_{\theta_2}\end{aligned} \tag{93}$$

$$\begin{aligned}[D] = [I_{\text{sc},B}] + \Big([I_{\text{S}_1,S_{c_1}}] + [I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] - \\ m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\Big)\hat{\boldsymbol{s}}_1\boldsymbol{B}_{\theta_1} + \Big([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\Big)\hat{\boldsymbol{s}}_2\boldsymbol{B}_{\theta_2}\end{aligned} \tag{94}$$

and vectors

$$\begin{gathered}\mathbf{v}_{\text{trans}} = \mathbf{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c} - \\ m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\mathbf{r}'_{S_c/B} - m_{\text{S}_2}\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} - [\tilde{\mathbf{r}}_{S_{c_2}/S_2}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1}\right) + \\ m_{\text{S}}[\tilde{\mathbf{r}}_{S_c/S_1}]\hat{\boldsymbol{s}}_1\boldsymbol{C}_{\theta_1} + m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\hat{\boldsymbol{s}}_2\boldsymbol{C}_{\theta_2}\end{gathered} \tag{95}$$

$$
\begin{aligned}
\mathbf{v}_{\text{rot}} = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \\
\left([I'_{\text{S}_1,S_{c_1}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_1,S_{c_1}}]\right)\boldsymbol{\omega}_{\mathcal{S}_1/\mathcal{B}} - \left([I'_{\text{S}_2,S_{c_2}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S}_2,S_{c_2}}]\right)\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{B}} - \\
\left([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}]\boldsymbol{\omega}_{\mathcal{S}_2/\mathcal{S}_1} - \\
m_{\text{S}_1}\left([\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_1}/B}]\right)\mathbf{r}'_{S_{c_1}/B} - \\
m_{\text{S}_2}\left([\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_1/\mathcal{B}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\mathbf{r}}_{S_{c_2}/B}]\right)\mathbf{r}'_{S_{c_2}/B} - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}_2/\mathcal{S}_1}]\mathbf{r}'_{S_{c_2}/S_2} - \\
\left([I_{\text{S}_1,S_{c_1}}] + [I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_1}[\tilde{\mathbf{r}}_{S_{c_1}/B}][\tilde{\mathbf{r}}_{S_{c_1}/S_1}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_1}]\right)\hat{\boldsymbol{s}}_1\boldsymbol{C}_{\theta_1} - \\
\left([I_{\text{S}_2,S_{c_2}}] - m_{\text{S}_2}[\tilde{\mathbf{r}}_{S_{c_2}/B}][\tilde{\mathbf{r}}_{S_{c_2}/S_2}]\right)\hat{\boldsymbol{s}}_2\boldsymbol{C}_{\theta_2}
\end{aligned}
\tag{96}
$$

**Validation**

The same validation tests are performed for the dual-axis rotating rigid body system, shown in Figures 9, 10, 11 and 12. As before, the angular momentum and energy quantities are conserved throughout the simulation, as only conservative forces and torques are acting on the spacecraft. This implies a high level of confidence that both the mathematical derivation and the software implementation are correct and follow fundamental physical principles.

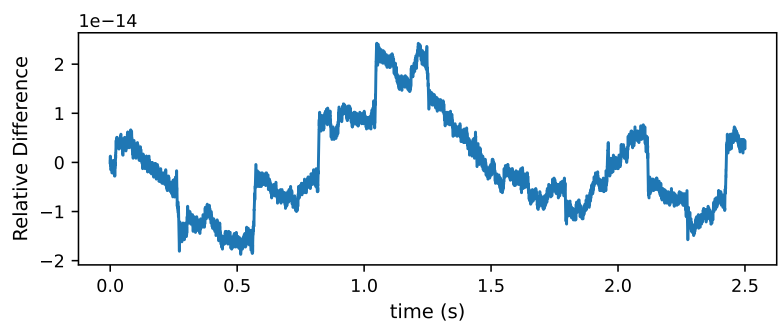


**Figure 9. Orbital energy using a dual-axis rotating rigid body.**

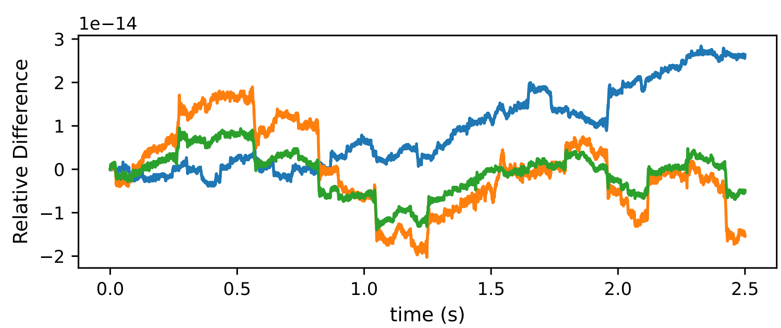


**Figure 10. Orbital angular momentum using a dual-axis rotating rigid body.**

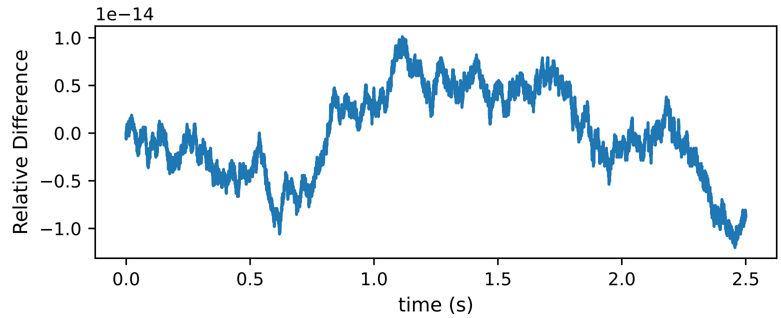


**Figure 11. Rotational energy using a dual-axis rotating rigid body.**

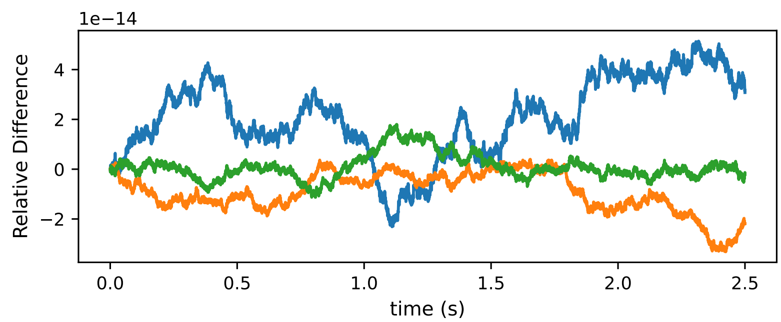


**Figure 12. Rotational angular momentum using a dual-axis rotating rigid body.**

## CONCLUSION

With spacecraft becoming more complex, there is a need for a robust simulation architecture that can replicate the spacecraft's behavior throughout its mission. Creating a general and modular representation of common categories of spacecraft components saves time and effort for the engineers while retaining the high fidelity needed to guarantee that the mission objectives are met. This work provides an architecture and the corresponding equations of motion for simulating single and dual-axis rotating rigid components in a general, modular way.

A dynamics architecture is proposed, where shared component classes are created to minimize code repetition and centralize the validation of the functions common to each class. Each abstract class can then represent specific components with their particular variables and methods while sharing the general equations of motion and kinematics contributions.

The equations of motion of the single and dual-axis are comprehensively derived without making any assumptions on the frame, spin axis, or the location of the center of mass. The outcome is a universal formulation of the equations that describe these components. Validation is done to both formulations by verifying energy and angular momentum conservation. It is shown that both models agree to these fundamental physical conservation laws when only conservative forces are present.

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