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Fully-Coupled Dynamical Jitter Modeling Of A Rigid Spacecraft With Imbalanced Double-Gimbal Variable-Speed Control Moment Gyros

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Abstract

Much attention has been paid to a double-gimbal variable-speed control moment gyro (DGVSCMG) as three-dimensional torque generator. However, since this device typically operate at high wheel speeds, mass imbalances within the wheels act as a primary source of angular jitter. An estimate of jitter amplitude may be found by modeling imbalance torques as external disturbance forces and torques on the spacecraft. In this case, mass imbalances are lumped into static and dynamic imbalance parameters, allowing jitter force and torque to be simply proportional to wheel speed squared. A physically realistic dynamic model may be obtained by defining mass imbalances in terms of a wheel center of mass location and inertia tensor. The fully-coupled dynamic model allows for momentum and energy validation of the system. This paper presents a generalized approach to DGVSCMG imbalance modeling of a rigid spacecraft hub with n DGVSCMGs. Through the numerical simulations, the effectiveness of the proposed dynamic model is demonstrated.

1. Introduction

Momentum exchange devices (MEDs) are used for an attitude control of a spacecraft. Recently, much attention has been paid to a double-gimbal variable-speed control moment gyro (DGVSCMG)[1–3] as a new type of MED of a spacecraft. A DGVSCMG has two gimbals attached to one variable speed wheel and can generate large three dimensional torques if the RW motor torque is sized accordingly. Implementing DGVSCMGs for attitude control can reduce the number of actuators and the total weight of actuators, which leads to reduced mass and volume within the spacecraft. On the other hand, a key source of pointing jitters are due to wheel or gimbals mass imbalance about the wheel spin axis or gimbal rotation axes in a DGVSCMG. Although these effects are often characterized through experimentation in order to validate requirements, it is of interest to include jitter in a computer simulation of the spacecraft in the early stages of spacecraft development. An estimation of jitter amplitude may be found by modeling wheel or gimbals imbalance torques as an external disturbance on the spacecraft. A physically realistic dynamic model may be obtained by defining mass imbalances in terms of a wheel or gimbals center of mass locations and inertia tensor. The fully-coupled dynamic model allows for momentum and energy validation of the system. This is often critical when modeling additional complex dynamical behavior such as flexible dynamics and fuel slosh. Furthermore, it is necessary to use the fully-coupled model in instances where the relative mass properties of the spacecraft with respect to the DGVSCMGs cause the simplified jitter model to be inaccurate.

Previous DGVSCMG studies[1–3] assumed the balanced DGVSCMG that ignored the mass imbalance of the gimbals or wheel and previous studies[4–8] put emphasis on empirical modeling of MED jitter and the effect of MED jitter within context of spacecraft flexible dynamics. Reference 9 presents a fully-coupled derivation of RW imbalance. It is demonstrated that the fully-coupled model allows an imbalanced RW to be simulated while still using momentum and energy tools for validation of the dynamics. Reference 10 discuss a fully-coupled model of single-gimbal CMG (SGCMG) imbalance, but present the results without a full derivation and fail to provide the complete system equations of motion. Furthermore, Reference 11 develops a fully-coupled derivation of single-gimbal variable-speed CMG (SGVSCMG) imbalance. This prior work demonstrates that the fully-coupled

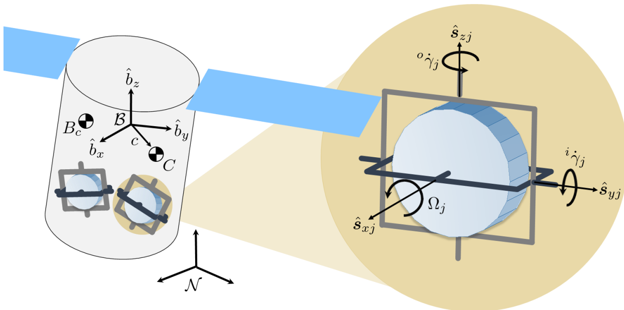

Fig. 1: Spacecraft model.

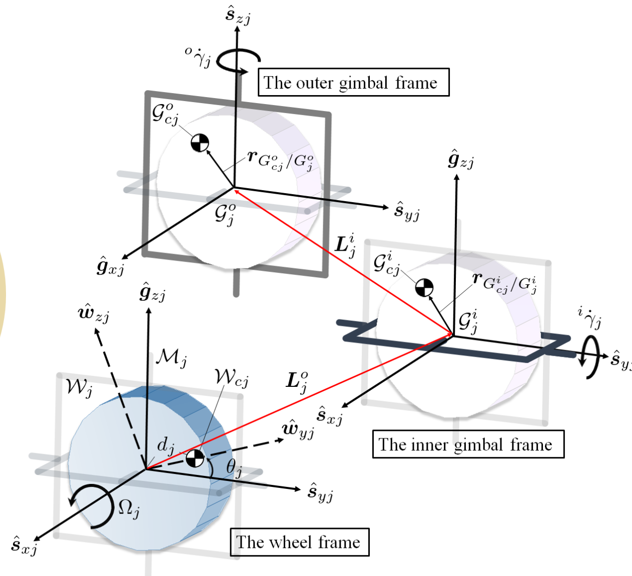

Fig. 2: j^{th} DGVSCMG frames and axes.

model allows an imbalanced VSCMG to be simulated. This paper presents a fully-coupled static and dynamic imbalance model of a rigid spacecraft equipped with n DGVSCMGs. This model is a generalized description of a fully-coupled static and dynamic imbalance modeling for a spacecraft with MEDs since the equation of motion (EOM) of DGVSCMGs includes the EOM of all MEDs such as RW, SGCMG or VSCMG. Jitter is also produced through the higher order structural resonances of the MED attachment mechanism and the bending modes of the rotors. The presented work, however, only considers the classical static and dynamic imbalance behaviors. While a general formulation of these imbalances, the resulting DGVSCMG equations of motion are very complex to develop and implement.

2. Problem Statement

The problem consists of modeling the static and dynamic imbalance of DGVSCMGs' assemblies attached to a rigid spacecraft as in Fig. 1. In order to develop the equations of motion in a general way, arbitrary locations, inertia tensors, and center of mass locations are considered for the spacecraft hub, inner/outer gimbals, and wheels. Additionally, the wheel center of mass is not assumed to lie on the gimbal axis of the DGVSCMG, and the wheel frame origin and gimbal frame origin are not assumed to coincide. Figure 2 shows the frame and variable definitions used for this problem.

2.1 Reference Frame Definitions

Figures 2 show the frame and variable definitions used for this problem. The formulation involves a rigid hub with its center of mass location labeled as point $\boldsymbol{B}_c$ and N DGVSCMGs with their center of mass in the outer gimbals, inner gimbals and wheels. The j^{th} DGVSCMG is labeled as the outer gimbal frame $\mathcal{G}^o_{cj}$, the inner gimbal frame $\mathcal{G}^i_{cj}$ and the wheel frame $\mathcal{W}_{cj}$, respectively. The inertial frame and the body frame is denoted $\mathcal{N}$ and $\mathcal{B}$. The basis vector of the body frame is defined as follows:

$$\mathcal{B} : \{B,\ \hat{\boldsymbol{b}}_x,\ \hat{\boldsymbol{b}}_y,\ \hat{\boldsymbol{b}}_z\}, \tag{1}$$

and

$$\mathcal{G}^o_j : \{G^o_j,\ \hat{\boldsymbol{g}}_{xj},\ \hat{\boldsymbol{s}}_{yj},\ \hat{\boldsymbol{s}}_{zj}\} \tag{2}$$
$$\mathcal{G}^i_j : \{G^i_j,\ \hat{\boldsymbol{s}}_{xj},\ \hat{\boldsymbol{s}}_{yj},\ \hat{\boldsymbol{g}}_{zj}\} \tag{3}$$
$$\mathcal{W}_j : \{W_j,\ \hat{\boldsymbol{s}}_{xj},\ \hat{\boldsymbol{w}}_{yj},\ \hat{\boldsymbol{w}}_{zj}\} \tag{4}$$
$$\mathcal{M}_j : \{W_j,\ \hat{\boldsymbol{s}}_{xj},\ \hat{\boldsymbol{s}}_{yj},\ \hat{\boldsymbol{g}}_{zj}\}. \tag{5}$$

Note that the basis vectors at the motor frame $\mathcal{M}_j$ of the j^{th} RW with the origin of coordinates W_j is same direction towards the basis vectors at $\mathcal{G}^i_j$ but the origin of coordinates is difference.

2.2 Variable Definitions

Parameters relating to the spacecraft hub are denoted with a subscript text B. Parameters relating to the j^{th} inner/outer gimbal and wheel are denoted with subscripts text G^i_j, G^o_j and W_j, respectively. The hub, inner/outer gimbal, and wheel each are allowed center of mass offsets from their respective coordinate frame origins. The spacecraft hub is the spacecraft structure without the DGSCMGs. The hub's center of mass location is labeled as B_c. This location is described with respect to the body frame origin as $\boldsymbol{r}_{B_c/B}$. The inner/outer gimbals are also allowed a general center of mass offset from the gimbal frame origins. This location are labeled as G^i_{cj}, G^o_{cj} and are located with respect to the inner/outer gimbal frame origins as $\boldsymbol{r}_{G^i_{cj}/G^i_j}$ and $\boldsymbol{r}_{G^o_{cj}/G^o_j}$. The wheel's center of mass location is labeled somewhat differently. The wheel center of mass is assumed to lie on the $\hat{\boldsymbol{w}}_{yj}$ axis a length d_j from the wheel frame origin. Three-dimensional offset vectors $\boldsymbol{L}^i_j = [L^i_{xj}, L^i_{yj}, L^i_{zj}]^T$, $\boldsymbol{L}^o_j = [L^o_{xj}, L^o_{yj}, L^o_{zj}]^T$ are introduce between frames not to result in loss of generality. Since the inner/outer gimbal and wheel centers of mass change with time, so does the overall spacecraft center of mass. The time-varying center of mass of the entire system is denoted $\boldsymbol{c}$.

3. Equation of Motion

3.1 Translational Motion

The derivation of the translational EOMs begins with Newton's second law for the center of mass of the spacecraft as follows:

$$\ddot{\boldsymbol{r}}_{C/N} = \frac{\boldsymbol{F}}{m_{sc}} \tag{6}$$

together with

$$m_{sc} = m_B + \sum_{j=1}^{N}(m_{G^o_j} + m_{G^i_j} + m_{W_j}) \tag{7}$$

The force vector $\boldsymbol{F}$ is the sum of the external forces on the spacecraft which has the spacecraft mass m_{sc}. Ultimately, the acceleration of the body frame or point B is desired, which is expressed through

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \ddot{\boldsymbol{c}}. \tag{8}$$

The center of mass $\boldsymbol{c}$ is time variant and it expressed as

$$\boldsymbol{c} = \frac{1}{m_{sc}}(m_B \boldsymbol{r}_{B_c/B} + \sum_{j=1}^{N}(m_{G^o_j}\boldsymbol{r}_{G^o_{cj}/B} + m_{G^i_j}\boldsymbol{r}_{G^i_{cj}/B} + m_{W_j}\boldsymbol{r}_{W_{cj}/B})). \tag{9}$$

Find the second inertial derivative of point $\boldsymbol{c}$.

$$\ddot{\boldsymbol{c}} = \boldsymbol{c}'' + \dot{\boldsymbol{\omega}} \times \boldsymbol{c} + 2\boldsymbol{\omega} \times \boldsymbol{c}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{c}). \tag{10}$$

Taking the second body frame time derivatives of $\boldsymbol{c}$ results in

$$\boldsymbol{c}'' = \frac{1}{m_{sc}} \sum_{j=1}^{N}(m_{G^o_j}\boldsymbol{r}''_{G^o_{cj}/B} + m_{G^i_j}\boldsymbol{r}''_{G^i_{cj}/B} + m_{W_j}\boldsymbol{r}''_{W_{cj}/B}) \tag{11}$$

where

$$\boldsymbol{r}''_{G^o_{cj}/B} = ({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj}) \times \boldsymbol{r}_{G^o_{cj}/G^o_j} + ({}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj}) \times \boldsymbol{r}'_{G^o_{cj}/B} \tag{12}$$

$$\begin{aligned}\boldsymbol{r}''_{G^i_{cj}/B} =& ({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} - {}^i\dot{\gamma}_j {}^o\dot{\gamma}_j \hat{\boldsymbol{g}}_{xj}) \times \boldsymbol{r}_{G^i_{cj}/G^i_j} \\ &+ ({}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{yj}) \times \boldsymbol{r}'_{G^i_{cj}/G^i_j} + (L^o_{xj}{}^o\ddot{\gamma}_j \\ &- L^o_{yj}{}^o\dot{\gamma}_j^2)\hat{\boldsymbol{s}}_{yj} - (L^o_{xj}{}^o\dot{\gamma}_j^2 + L^o_{yj}{}^o\ddot{\gamma}_j)\hat{\boldsymbol{g}}_{xj}\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{r}''_{W_{cj}/B} =& \Big(d_j({}^i\ddot{\gamma}_j \mathrm{s}\theta_j - {}^o\ddot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j + 2\Omega_j {}^i\dot{\gamma}_j \mathrm{c}\theta_j \\ &+ 2\Omega_j {}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + 2{}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{s}^i\gamma_j \mathrm{c}\theta_j + {}^o\dot{\gamma}_j^2 \mathrm{s}^i\gamma_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j) \\ &- L^i_{xj}({}^o\dot{\gamma}_j^2 \mathrm{c}^{2i}\gamma_j + {}^i\dot{\gamma}_j^2) + L^i_{yj}({}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{s}^i\gamma_j - {}^o\ddot{\gamma}_j \mathrm{c}^i\gamma_j) \\ &+ L^i_{zj}({}^o\dot{\gamma}_j^2 \mathrm{s}^i\gamma_j \mathrm{c}^i\gamma_j + \Omega_j {}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j + {}^i\ddot{\gamma}_j) \\ &- L^o_{xj}{}^o\dot{\gamma}_j^2 \mathrm{c}^{2i}\gamma_j + L^o_{yj}(2{}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{s}^i\gamma_j - {}^o\ddot{\gamma}_j \mathrm{c}^i\gamma_j)\Big)\hat{\boldsymbol{s}}_{xj} \\ &+ \Big(-d_j({}^o\dot{\gamma}_j^2 \mathrm{c}\theta_j + {}^o\ddot{\gamma}_j \mathrm{s}^i\gamma_j \mathrm{s}\theta_j + \dot{\Omega}_j \mathrm{s}\theta_j + \Omega_j {}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j \mathrm{c}\theta_j) \\ &+ L^i_{xj}{}^o\ddot{\gamma}_j \mathrm{c}^i\gamma_j - L^i_{yj}{}^o\dot{\gamma}_j^2 - L^i_{zj}({}^o\ddot{\gamma}_j \mathrm{s}^i\gamma_j + \dot{\Omega}_j) + L^o_{xj}{}^o\ddot{\gamma}_j \\ &- L^o_{yj}{}^o\dot{\gamma}_j^2\Big)\hat{\boldsymbol{s}}_{yj} + \Big(d_j({}^o\ddot{\gamma}_j \mathrm{s}^i\gamma_j \mathrm{c}\theta_j - {}^o\dot{\gamma}_j^2 \mathrm{s}^{2i}\gamma_j \mathrm{s}\theta_j \\ &- {}^i\dot{\gamma}_j^2 \mathrm{s}\theta_j + \dot{\Omega}_j \mathrm{c}\theta_j + 2{}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j - 2{}^o\dot{\gamma}_j \Omega_j \mathrm{s}^i\gamma_j \mathrm{s}\theta_j \\ &- \Omega_j^2 \mathrm{s}\theta_j) + L^i_{xj}({}^o\dot{\gamma}_j^2 \mathrm{c}^i\gamma_j \mathrm{s}^i\gamma_j - {}^i\ddot{\gamma}_j) + L^i_{yj}({}^o\ddot{\gamma}_j \mathrm{s}^i\gamma_j \\ &+ 2{}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{c}^i\gamma_j) - L^i_{zj}({}^i\dot{\gamma}_j^2 + {}^o\dot{\gamma}_j^2 \mathrm{s}^{2i}\gamma_j + {}^o\dot{\gamma}_j \Omega_j \mathrm{s}^i\gamma_j) \\ &+ L^o_{xj}{}^o\dot{\gamma}_j^2 \mathrm{s}^i\gamma_j + L^o_{yj}({}^o\ddot{\gamma}_j \mathrm{s}^i\gamma_j + 2{}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \mathrm{c}^i\gamma_j)\Big)\hat{\boldsymbol{g}}_{zj}\end{aligned} \tag{14}$$

Noting that $\mathrm{s} = \sin$, $\mathrm{c} = \cos$. Substituting Eq. (10) into Eq. (8), it is given by

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \boldsymbol{c}'' + [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{c}. \tag{15}$$

Substituting Eq. (11) into Eq. (15) and group second-order terms to obtain the translational equations of motion.

$$\begin{aligned}\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} + \frac{1}{m_{sc}}\sum_{j=1}^{N}\Big(\boldsymbol{f}_{oj}{}^o\ddot{\gamma}_j + \boldsymbol{f}_{ij}{}^i\ddot{\gamma}_j + \boldsymbol{f}_{wj}\dot{\Omega}_j\Big) \\ = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{c} - \frac{1}{m_{sc}}\sum_{j=1}^{N}\Big(\tilde{\boldsymbol{f}}_j\Big)\end{aligned} \tag{16}$$

where

$$\begin{aligned}\boldsymbol{f}_{oj} =& m_{G^o_j}\Big(\hat{\boldsymbol{s}}_{zj} \times \boldsymbol{r}_{G^o_{cj}/G^o_j}\Big) + m_{G^i_j}\Big(\hat{\boldsymbol{s}}_{zj} \times \boldsymbol{r}_{G^i_{cj}/G^i_j} \\ &+ L^o_{xj}\hat{\boldsymbol{s}}_{yj} + L^o_{yj}\hat{\boldsymbol{g}}_{xj}\Big) + m_{W_j}\big((-d_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j \\ &- L^i_{yj}\mathrm{c}^i\gamma_j - L^o_{yj}\mathrm{c}^i\gamma_j)\hat{\boldsymbol{s}}_{xj} + (-d_j \mathrm{s}^i\gamma_j \mathrm{s}\theta_j \\ &+ L^i_{xj}\mathrm{c}^i\gamma_j - L^i_{zj}\mathrm{s}^i\gamma_j + L^o_{xj})\hat{\boldsymbol{s}}_{yj} + (d_j \mathrm{s}^i\gamma_j \mathrm{c}\theta_j \\ &+ L^i_{yj}\mathrm{s}^i\gamma_j + L^o_{yj}\mathrm{s}^i\gamma_j)\hat{\boldsymbol{g}}_{zj}\big) \\ \boldsymbol{f}_{ij} =& m_{G^i_j}\Big(\hat{\boldsymbol{s}}_{yj} \times \boldsymbol{r}_{G^i_{cj}/G^i_j}\Big) + m_{W_j}\big((d_j \mathrm{s}\theta_j + L^i_{zj})\hat{\boldsymbol{s}}_{xj}\big) \\ \boldsymbol{f}_{wj} =& m_{W_j}\big(-(d_j \mathrm{s}\theta_j + L^i_{zj})\hat{\boldsymbol{s}}_{yj} + d_j \mathrm{c}\theta_j \hat{\boldsymbol{g}}_{zj}\big)\end{aligned}$$

and

$$\begin{aligned}\tilde{\boldsymbol{f}}_j &= m_{G_j^o}\left(({}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj})\times\boldsymbol{r}'_{G^o_{cj}/B}\right)\\&+m_{G_j^i}\Big(-({}^i\dot{\gamma}_j{}^o\dot{\gamma}_j\hat{\boldsymbol{g}}_{xj})\times\boldsymbol{r}_{G^i_{cj}/G^i_j}+({}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj}\\&+{}^i\dot{\gamma}_j\hat{\boldsymbol{s}}_{yj})\times\boldsymbol{r}'_{G^i_{cj}/G^i_j}-(L^o_{yj}\hat{\boldsymbol{s}}_{yj}+L^o_{xj}\hat{\boldsymbol{g}}_{xj}){}^o\dot{\gamma}_j^2\Big)\\&+m_{W_j}\big(d_j(2\Omega_j{}^i\dot{\gamma}_j\mathrm{c}\theta_j+2\Omega_j{}^o\dot{\gamma}_j\mathrm{c}^i\gamma_j\mathrm{s}\theta_j\\&+2{}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{s}^i\gamma_j\mathrm{c}\theta_j+{}^o\dot{\gamma}_j^2\mathrm{s}^i\gamma_j\mathrm{c}^i\gamma_j\mathrm{s}\theta_j)-L^i_{xj}({}^o\dot{\gamma}_j^2\mathrm{c}^{2i}\gamma_j\\&+{}^i\dot{\gamma}_j^2)+L^i_{yj}({}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{s}^i\gamma_j)+L^i_{zj}({}^o\dot{\gamma}_j^2\mathrm{s}^i\gamma_j\mathrm{c}^i\gamma_j\\&+\Omega_j{}^o\dot{\gamma}_j\mathrm{c}^i\gamma_j)-L^o_{xj}{}^o\dot{\gamma}_j^2\mathrm{c}^{2i}\gamma_j+2L^o_{yj}{}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{s}^i\gamma_j\Big)\,\hat{\boldsymbol{s}}_{xj}\\&+\Big(-d_j({}^o\dot{\gamma}_j^2\mathrm{c}\theta_j+\Omega_j{}^o\dot{\gamma}_j\mathrm{s}^i\gamma_j\mathrm{c}\theta_j)-L^i_{yj}{}^o\dot{\gamma}_j^2\\&-L^o_{yj}{}^o\dot{\gamma}_j^2\Big)\,\hat{\boldsymbol{s}}_{yj}+\Big(d_j(-{}^o\dot{\gamma}_j^2\mathrm{s}^{2i}\gamma_j\mathrm{s}\theta_j-{}^i\dot{\gamma}_j^2\mathrm{s}\theta_j\\&+2{}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{c}^i\gamma_j\mathrm{c}\theta_j-2{}^o\dot{\gamma}_j\Omega_j\mathrm{s}^i\gamma_j\mathrm{s}\theta_j-\Omega_j^2\mathrm{s}\theta_j)\\&+L^i_{xj}({}^o\dot{\gamma}_j^2\mathrm{c}^i\gamma_j\mathrm{s}^i\gamma_j-{}^i\ddot{\gamma}_j)+2L^i_{yj}{}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{c}^i\gamma_j\\&-L^i_{zj}({}^i\dot{\gamma}_j^2+{}^o\dot{\gamma}_j^2\mathrm{s}^{2i}\gamma_j+{}^o\dot{\gamma}_j\Omega_j\mathrm{s}^i\gamma_j)+L^o_{xj}{}^o\dot{\gamma}_j^2\mathrm{s}^i\gamma_j\\&+2L^o_{yj}{}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\mathrm{c}^i\gamma_j\big)\,\hat{\boldsymbol{g}}_{zj}\end{aligned}\tag{17}$$

This equation represents 3 DOFs and contains all second order states ($\ddot{\boldsymbol{r}}_{B/N}$, $\dot{\boldsymbol{\omega}}$, ${}^o\ddot{\gamma}$, ${}^i\ddot{\gamma}$, $\dot{\Omega}$). Removing wheel imbalance terms and assuming a symmetrical DGVSCMG (i.e. $\boldsymbol{r}_{G^o_{cj}/G^o_j}=\boldsymbol{0}$, $\boldsymbol{r}_{G^i_{cj}/G^i_j}=\boldsymbol{0}$, $L^o_{xj}=0$, $L^o_{yj}=0$, $L^o_{zj}=0$, $L^i_{xj}=0$, $L^i_{yj}=0$, $L^i_{zj}=0$, $d_j=0$) gives the following equation.

$$m_{sc}\ddot{\boldsymbol{r}}_{B/N}-m_{sc}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}=\boldsymbol{F}-2m_{sc}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}'-m_{sc}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c}\tag{18}$$

where

$$\boldsymbol{F}=\ddot{\boldsymbol{r}}_{C/N}-\sum_{j=1}^{N}\left(({}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj})\times\boldsymbol{r}'_{G^o_{cj}/B}\right)\tag{19}$$

Thus, the balanced DGVSCMG translational equation of motion does not certain any second order terms relating to the wheel or inner/outer gimbal. The following section shows the derivation of the rotational equations of motion.

3.2 Rotational Motion

The derivation of rotational EOMs starts with the angular momentum of the spacecraft about point B.

$$\boldsymbol{H}_{sc,B}=\boldsymbol{H}_{\mathrm{B},B}+\sum_{j=1}^{N}(\boldsymbol{H}_{G^o_j,B}+\boldsymbol{H}_{G^i_j,B}+\boldsymbol{H}_{W_j,B})\tag{20}$$

The inertial time derivative of angular momentum when the body fixed coordinate frame origin is not coincident with the center of mass of the body is

$$\dot{\boldsymbol{H}}_{sc,B}=\boldsymbol{L}_B+m_{sc}\ddot{\boldsymbol{r}}_{B/N}\times\boldsymbol{c},\tag{21}$$

where $\boldsymbol{L}_B$ is the vector sum of external torque acting on the spacecraft. Differentiating Eq. (20), the inertial derivative of the spacecraft angular momentum is expressed as

$$\dot{\boldsymbol{H}}_{sc,B}=\dot{\boldsymbol{H}}_{\mathrm{B},B}+\sum_{j=1}^{N}(\dot{\boldsymbol{H}}_{G^o_j,B}+\dot{\boldsymbol{H}}_{G^i_j,B}+\dot{\boldsymbol{H}}_{W_j,B})\tag{22}$$

Thus, in order to use Eq. (21), each derivative on the right-hand side of Eq. (22) needs to be evaluated.

3.2.1 Derivative of the Hub Angular Momentum

The first step is to derive the hub angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{B},B}$. The hub angular momentum about point B_c is given by

$$\boldsymbol{H}_{\mathrm{B},B_c}=[\boldsymbol{I}_{\mathrm{B},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}.\tag{23}$$

Angular momentum about point B_c is related to point B using the following equation.

$$\boldsymbol{H}_{\mathrm{B},B}=\boldsymbol{H}_{\mathrm{B},B_c}+m_{\mathrm{B}}\boldsymbol{r}_{B_c/B}\times\dot{\boldsymbol{r}}_{B_c/B}\tag{24}$$

Taking the inertial time derivative of the hub's angular momentum yields

$$\dot{\boldsymbol{H}}_{\mathrm{B},B}=[\boldsymbol{I}_{\mathrm{B},B_c}]\dot{\boldsymbol{\omega}}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{B},B_c}]\boldsymbol{\omega}+m_{\mathrm{B}}\boldsymbol{r}_{B_c/B}\times\ddot{\boldsymbol{r}}_{B_c/B}\tag{25}$$

Note that the body rate pseudovector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ is abbreviated as $\boldsymbol{\omega}$ henceforth. Knowing that $\boldsymbol{r}_{B_c/B}$ is fixed with respect to the body frame, the following is defined

$$\ddot{\boldsymbol{r}}_{B_c/B}=\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{B_c/B}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{B_c/B})\tag{26}$$

Substituting Eq. (26) into Eq. (25) yields

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{B},B}&=[\boldsymbol{I}_{\mathrm{B},B_c}]\dot{\boldsymbol{\omega}}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{B},B_c}]\boldsymbol{\omega}\\&+m_{\mathrm{B}}\boldsymbol{r}_{B_c/B}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{B_c/B})\\&+m_{\mathrm{B}}\boldsymbol{r}_{B_c/B}\times(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{B_c/B}))\end{aligned}\tag{27}$$

Employing the Jacobi triple-product identity, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})$, on the right-hand side of Eq. (27)

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{B},B}&=[\boldsymbol{I}_{\mathrm{B},B_c}]\dot{\boldsymbol{\omega}}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{B},B_c}]\boldsymbol{\omega}\\&+m_{\mathrm{B}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\dot{\boldsymbol{\omega}}\\&+m_{\mathrm{B}}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\boldsymbol{\omega}\end{aligned}\tag{28}$$

The parallel axis theorem relates inertia about the hub center of B_c to the hub origin B.

$$[\boldsymbol{I}_{\mathrm{B},B}]=[\boldsymbol{I}_{\mathrm{B},B_c}]+m_{\mathrm{B}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\tag{29}$$

The hub angular momentum derivative simplifies to

$$\dot{\boldsymbol{H}}_{\mathrm{B},B}=[\boldsymbol{I}_{\mathrm{B},B}]\dot{\boldsymbol{\omega}}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{B},B}]\boldsymbol{\omega}.\tag{30}$$

3.2.2 Derivative of the outer gimbal angular momentum

The next step is to derive the outer gimbal angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{G}_j^o,B}$. The angular velocity of the outer gimbal frame with respect to inertial is

$$\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{B}} = \boldsymbol{\omega} + {}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \tag{31}$$

The outer gimbal angular momentum about point G_{cj}^o is given by

$$\begin{aligned}\boldsymbol{H}_{G_j^o/G_{cj}^o} &= [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} \\ &= [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}](\boldsymbol{\omega} + {}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj}).\end{aligned} \tag{32}$$

Angular momentum about point G_{cj}^o is related to point B using the following equation.

$$\boldsymbol{H}_{G_j^o/B} = \boldsymbol{H}_{G_j^o/G_{cj}^o} + m_{G_j^o}\boldsymbol{r}_{G_{cj}^o/B} \times \dot{\boldsymbol{r}}_{G_{cj}^o/B} \tag{33}$$

Taking the inertial derivative yields

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{G}_j^o,B} &= [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj}) + [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]'\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} + m_{\mathrm{G}_j^o}\boldsymbol{r}_{G_{cj}^o/B} \times \ddot{\boldsymbol{r}}_{G_{cj}^o/B}\end{aligned} \tag{34}$$

The outer gimbal inertia tensor about the outer gimbal center of mass $[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]$ and its body frame derivative $[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]'$ are defined. Expressed in the outer gimbal frame,

$$[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}] = {}^{\mathcal{G}_j^o}\begin{bmatrix} I_{\mathrm{G}_{11j}^o} & I_{\mathrm{G}_{12j}^o} & I_{\mathrm{G}_{13j}^o} \\ I_{\mathrm{G}_{12j}^o} & I_{\mathrm{G}_{22j}^o} & I_{\mathrm{G}_{23j}^o} \\ I_{\mathrm{G}_{13j}^o} & I_{\mathrm{G}_{23j}^o} & I_{\mathrm{G}_{33j}^o}\end{bmatrix} \tag{35}$$

By expressing this tensor in a frame independent form, the body frame derivative is found to be,

$$[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]' = {}^o\dot{\gamma}_j \; {}^{\mathcal{G}_j^o}\begin{bmatrix} -2I_{\mathrm{G}_{12j}^o} & I_{12} & -I_{\mathrm{G}_{23j}^o} \\ I_{12} & 2I_{\mathrm{G}_{12j}^o} & I_{\mathrm{G}_{13j}^o} \\ -I_{\mathrm{G}_{23j}^o} & I_{\mathrm{G}_{13j}^o} & 0\end{bmatrix} \tag{36}$$

where

$$I_{12} = I_{\mathrm{G}_{11j}^o} - I_{\mathrm{G}_{22j}^o} \tag{37}$$

The second inertial derivative of $\boldsymbol{r}_{G_{cj}^o/B}$ is given by

$$\begin{aligned}\ddot{\boldsymbol{r}}_{G_{cj}^o/B} &= {}^o\ddot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}_{G_{cj}^o/G_j^o} + ({}^o\dot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}'_{G_{cj}^o/G_j^o} \\ &+ [\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]^T\dot{\boldsymbol{\omega}} + 2\times[\tilde{\boldsymbol{r}}'_{G_{cj}^o/B}]^T\boldsymbol{\omega} \\ &+ [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{cj}^o/B}\end{aligned} \tag{38}$$

Note that $\boldsymbol{r}''_{G_{cj}^o/B}$ is given by Eq. (12). Substituting this into Eq. (34) yields

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{G}_j^o,B} &= [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj}) + [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]'\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} + m_{\mathrm{G}_j^o}[\tilde{\boldsymbol{r}}_{G_{cj}/B}] \\ &\Big[{}^o\ddot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}_{G_{cj}^o/G_j^o} + {}^o\dot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}'_{G_{cj}^o/G_j^o} + [\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]^T\dot{\boldsymbol{\omega}} \\ &+2[\tilde{\boldsymbol{r}}'_{G_{cj}^o/B}]^T\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{cj}^o/B}\Big]\end{aligned} \tag{39}$$

The parallel axis theorem relating the outer gimbal inertia about point B to the outer gimbal inertia about point G_{cj}^o is given by

$$[\boldsymbol{I}_{\mathrm{G}_j^o,B}] = [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}] + m_{G_j^o}[\tilde{\boldsymbol{r}}_{G_{cj}^o/B}][\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]^T \tag{40}$$

Using Eq. (40), Eq. (39) simplifies to

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{G}_j^o,B} &= [\boldsymbol{I}_{\mathrm{G}_j^o,B}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]{}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \\ &+[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]'\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,B}]\boldsymbol{\omega}+[\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]{}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \\ &+ m_{\mathrm{G}_j^o}[\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]\Big[{}^o\ddot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}_{G_{cj}^o/G_j^o} + {}^o\dot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}'_{G_{cj}^o/G_j^o} \\ &+ 2[\tilde{\boldsymbol{r}}'_{G_{cj}^o/B}]^T\boldsymbol{\omega}\Big]\end{aligned} \tag{41}$$

Employing the body frame derivative of the parallel axis theorem.

$$\begin{aligned}[\boldsymbol{I}_{\mathrm{G}_j^o,B}]' =&[\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]' + m_{G_j^o}[\tilde{\boldsymbol{r}}'_{G_{cj}^o/B}][\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]^T \\ &+ m_{G_j^o}[\tilde{\boldsymbol{r}}_{G_{cj}^o/B}][\tilde{\boldsymbol{r}}'_{G_{cj}^o/B}]^T\end{aligned} \tag{42}$$

Eq. (41) is further simplified using Eq. (42) to give the outer gimbal angular momentum derivative.

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{G}_j^o,B} &= [\boldsymbol{I}_{\mathrm{G}_j^o,B}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{\mathrm{G}_j^o,B}]'\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,B}]\boldsymbol{\omega} \\ &+ [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]{}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + [\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]'{}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^o,G_{cj}^o}]{}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + m_{\mathrm{G}_j^o}[\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]\Big[{}^o\ddot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}_{G_{cj}^o/G_j^o} \\ &+{}^o\dot{\gamma}_j[\tilde{\hat{\boldsymbol{s}}}_{zj}]\boldsymbol{r}'_{G_{cj}^o/G_j^o}\Big] + m_{\mathrm{G}_j^o}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{cj}^o/B}]\boldsymbol{r}'_{G_{cj}^o/B}\end{aligned} \tag{43}$$

3.2.3 Derivative of the Inner Gimbal Angular Momentum

The next step is to derive the inner gimbal angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B}$. The angular velocity of the inner gimbal frame with respect to inertial is

$$\begin{aligned}\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} =&\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{G}_j^o} \\ =&\boldsymbol{\omega} + {}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j\hat{\boldsymbol{s}}_{yj}.\end{aligned} \tag{44}$$

The inner gimbal angular momentum about point G_{cj}^i is given by

$$\begin{aligned}\boldsymbol{H}_{\mathrm{G}_j^i,G_{cj}^i} &= [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \\ &= [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}](\boldsymbol{\omega} + {}^o\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j\hat{\boldsymbol{s}}_{yj}).\end{aligned} \tag{45}$$

Angular momentum about point G_{cj}^i is related to point B using the following equation.

$$\boldsymbol{H}_{\mathrm{G}_j^i,B} = \boldsymbol{H}_{\mathrm{G}_j^i,G_{cj}^i} + m_{G_j^i}\boldsymbol{r}_{G_{cj}^i/B} \times \dot{\boldsymbol{r}}_{G_{cj}^i/B}. \tag{46}$$

Take the inertial derivative.

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B} =&[\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j\hat{\boldsymbol{s}}_{yj} - {}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\hat{\boldsymbol{g}}_{xj}) \\ &+ [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]'\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \\ &+ m_{\mathrm{G}_j^i}\boldsymbol{r}_{G_{cj}^i/B} \times \ddot{\boldsymbol{r}}_{G_{cj}^i/B}\end{aligned} \tag{47}$$

The inner gimbal inertia tensor about the gimbal center of mass $[\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]$ and its body frame derivative $[\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]'$ are defined. Expressed in the inner gimbal frame,

$$[\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}] = {}^{\mathcal{G}_j^i}\begin{bmatrix} I_{\mathrm{G}_{11j}^i} & I_{\mathrm{G}_{12j}^i} & I_{\mathrm{G}_{13j}^i} \\ I_{\mathrm{G}_{12j}^i} & I_{\mathrm{G}_{22j}^i} & I_{\mathrm{G}_{23j}^i} \\ I_{\mathrm{G}_{13j}^i} & I_{\mathrm{G}_{23j}^i} & I_{\mathrm{G}_{33j}^i} \end{bmatrix} \quad (48)$$

By expressing this tensor in a frame independent form, the body frame derivative is found to be,

$$[\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]' = {}^{\mathcal{G}_j^i}\begin{bmatrix} 2\tilde{I}_{\mathrm{G}_{11j}^i} & \tilde{I}_{\mathrm{G}_{12j}^i} & \tilde{I}_{\mathrm{G}_{13j}^i} \\ \tilde{I}_{\mathrm{G}_{12j}^i} & 2\tilde{I}_{\mathrm{G}_{22j}^i} & \tilde{I}_{\mathrm{G}_{23j}^i} \\ \tilde{I}_{\mathrm{G}_{13j}^i} & \tilde{I}_{\mathrm{G}_{23j}^i} & 2\tilde{I}_{\mathrm{G}_{33j}^i} \end{bmatrix} \quad (49)$$

together with the principal axes of inertia

$$\begin{aligned} \tilde{I}_{\mathrm{G}_{11j}^i} &= I_{\mathrm{G}_{13j}^i}\,{}^i\dot{\gamma}_j - I_{\mathrm{G}_{12j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j \\ \tilde{I}_{\mathrm{G}_{22j}^i} &= I_{\mathrm{G}_{12j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j - I_{\mathrm{G}_{23j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j \\ \tilde{I}_{\mathrm{G}_{33j}^i} &= I_{\mathrm{G}_{23j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j - I_{\mathrm{G}_{13j}^i}\,{}^i\dot{\gamma}_j \end{aligned}$$

and the products of inertia

$$\begin{aligned} \tilde{I}_{\mathrm{G}_{12j}^i} =& I_{\mathrm{G}_{11j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j - I_{\mathrm{G}_{13j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j \\ & - I_{\mathrm{G}_{22j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j + I_{\mathrm{G}_{23j}^i}\,{}^i\dot{\gamma}_j \quad (50) \\ \tilde{I}_{\mathrm{G}_{13j}^i} =& I_{\mathrm{G}_{11j}^i}\,{}^i\dot{\gamma}_j + I_{\mathrm{G}_{12j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j \\ & - I_{\mathrm{G}_{23j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j + I_{\mathrm{G}_{33j}^i}\,{}^i\dot{\gamma}_j \quad (51) \\ \tilde{I}_{\mathrm{G}_{23j}^i} =& - I_{\mathrm{G}_{12j}^i}\,{}^i\dot{\gamma}_j + I_{\mathrm{G}_{13j}^i}\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j \\ & + I_{\mathrm{G}_{22j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j - I_{\mathrm{G}_{33j}^i}\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j. \quad (52) \end{aligned}$$

The second inertial derivative of $\boldsymbol{r}_{G_{cj}^i/B}$ is given by

$$\begin{aligned} \ddot{\boldsymbol{r}}_{G_{cj}^i/B} =& \boldsymbol{r}''_{G_{cj}^i/B} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{cj}^i/B} + \boldsymbol{\omega} \times \boldsymbol{r}'_{G_{cj}^i/B} \\ & + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{G_{cj}^i/B} \\ =& ({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} - {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \hat{\boldsymbol{g}}_{xj}) \times \boldsymbol{r}_{G_{cj}^i/G_j^i} \\ & + ({}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{yj}) \times \boldsymbol{r}'_{G_{cj}^i/G_j^i} + (L_{xj}^o\,{}^o\ddot{\gamma}_j \\ & - L_{yj}^o\,{}^o\dot{\gamma}_j^2)\hat{\boldsymbol{s}}_{yj} - (L_{xj}^o\,{}^o\dot{\gamma}_j^2 + L_{yj}^o\,{}^o\ddot{\gamma}_j)\hat{\boldsymbol{g}}_{xj} \\ & + [\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]^T\dot{\boldsymbol{\omega}} + 2 \times [\tilde{\boldsymbol{r}}'_{G_{cj}^i/B}]^T\boldsymbol{\omega} \\ & + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{cj}^i/B} \quad (53) \end{aligned}$$

Note that $\boldsymbol{r}''_{G_{cj}^i/B}$ is given by Eq. (13). Substituting this into Eq. (47) yields

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B} =& [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} \\ & + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j \hat{\boldsymbol{g}}_{zj} + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j \hat{\boldsymbol{g}}_{xj}) \\ & + [\boldsymbol{I}_{\mathrm{G}_j,G_{cj}^i}]'\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \\ & + m_{\mathrm{G}_j^i}[\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]\Big[\boldsymbol{r}''_{G_{cj}^i/B} + [\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]^T\dot{\boldsymbol{\omega}} \\ & + 2[\tilde{\boldsymbol{r}}'_{G_{cj}^i/B}]^T\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{cj}^i/B}\Big] \quad (54) \end{aligned}$$

The parallel axis theorem relating the outer gimbal inertia about point B to the outer gimbal inertia about point G_{cj}^i is given by

$$[\boldsymbol{I}_{\mathrm{G}_j^i,B}] = [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}] + m_{G_j^i}[\tilde{\boldsymbol{r}}_{G_{cj}^i/B}][\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]^T \quad (55)$$

Using Eq. (54), Eq. (55) simplifies to

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B} =& [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} \\ & + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \cos{}^i\gamma_j \hat{\boldsymbol{g}}_{zj} + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \sin{}^i\gamma_j \hat{\boldsymbol{g}}_{xj}) + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,B}]\boldsymbol{\omega} \\ & + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} + [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]'\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \\ & + m_{\mathrm{G}_j^i}[\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]\Big[\boldsymbol{r}''_{G_{cj}^i/B} + 2[\tilde{\boldsymbol{r}}'_{G_{cj}^i/B}]^T\boldsymbol{\omega}\Big] \quad (56) \end{aligned}$$

Employing the body frame derivative of the parallel axis theorem.

$$\begin{aligned} [\boldsymbol{I}_{\mathrm{G}_j^i,B}]' =& [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]' + m_{G_j^i}[\tilde{\boldsymbol{r}}'_{G_{cj}^i/B}][\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]^T \\ & + m_{G_j^i}[\tilde{\boldsymbol{r}}_{G_{cj}^i/B}][\tilde{\boldsymbol{r}}'_{G_{cj}^i/B}]^T \quad (57) \end{aligned}$$

Eq. (56) is further simplified using Eq. (57) to give the outer gimbal angular momentum derivative.

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B} =& [\boldsymbol{I}_{\mathrm{G}_j^i,B}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{\mathrm{G}_j^i,B}]'\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,B}]\boldsymbol{\omega} \\ & + [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j\,{}^i\hat{\boldsymbol{s}}_{yj} \\ & + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \hat{\boldsymbol{g}}_{zj} + {}^i\dot{\gamma}_j\,{}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j \hat{\boldsymbol{g}}_{xj}) \\ & + [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{B}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{B}} \\ & + m_{\mathrm{G}_j^i}[\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]\boldsymbol{r}''_{G_{cj}^i/B} + m_{\mathrm{G}_j^i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{cj}^i/B}]\boldsymbol{r}'_{G_{cj}^i/B} \quad (58) \end{aligned}$$

3.2.4 Derivative of the Wheel Angular Momentum

The next step is to derive the wheel angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{W}_j,B}$. The angular velocity of the wheel with respect to inertial is

$$\begin{aligned} \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} =& \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{G}_j^o} + \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{G}_j^i} \\ =& \boldsymbol{\omega} + {}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j\,{}^i\hat{\boldsymbol{s}}_{yj} + \Omega_j\,{}^w\hat{\boldsymbol{s}}_{xj} \quad (59) \end{aligned}$$

The wheel angular momentum about point W_{cj} is given by

$$\begin{aligned} \boldsymbol{H}_{\mathrm{W}_j/W_{cj}} =& [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \\ =& [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}](\boldsymbol{\omega} + {}^o\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + \Omega_j \hat{\boldsymbol{s}}_{xj}) \quad (60) \end{aligned}$$

Angular momentum about point W_{cj} is related to point B using the following equation.

$$\boldsymbol{H}_{\mathrm{W}_j/B} = \boldsymbol{H}_{\mathrm{W}_j/W_{cj}} + m_{W_j}\boldsymbol{r}_{W_{cj}/B} \times \dot{\boldsymbol{r}}_{W_{cj}/B} \quad (61)$$

Taking the inertial derivative yields

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,B} =& [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j \hat{\boldsymbol{s}}_{xj} \\ & + {}^o\dot{\gamma}_j\,{}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} \\ & + {}^i\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]'\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \\ & + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} + m_{\mathrm{W}_j}\boldsymbol{r}_{W_{cj}/B} \times \ddot{\boldsymbol{r}}_{W_{cj}/B} \quad (62) \end{aligned}$$

The wheel inertia tensor about the wheel center of mass $[\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]$ and its body frame derivative $[\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]'$ need to be defined. For this general RW model, the inertia matrix of the RW in the $\mathcal{W}_j$ frame is defined as

$$[\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}] = {}^{\mathcal{W}_j}\begin{bmatrix} J_{11j} & J_{12j} & J_{13j} \\ J_{12j} & J_{22j} & J_{23j} \\ J_{13j} & J_{23j} & J_{33j} \end{bmatrix} \tag{63}$$

By expressing this tensor in a frame independent form, the body frame derivative is found to be,

$$[\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' = {}^{\mathcal{W}_j}\begin{bmatrix} 2\tilde{J}_{11j} & \tilde{J}_{12j} & \tilde{J}_{13j} \\ \tilde{J}_{12j} & 2\tilde{J}_{22j} & \tilde{J}_{23j} \\ \tilde{J}_{13j} & \tilde{J}_{23j} & 2\tilde{J}_{33j} \end{bmatrix} \tag{64}$$

together with the principal axes of inertia

$$\begin{aligned} \tilde{J}_{11j} =& J_{12j}({}^i\dot{\gamma}_j \mathrm{s}\theta_j - {}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j) + J_{13j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j \\ &+ {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ \tilde{J}_{22j} =& J_{12j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j - {}^i\dot{\gamma}_j \mathrm{s}\theta_j) + J_{23j}({}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j + \Omega_j) \\ \tilde{J}_{33j} =& -J_{13j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ &+ J_{23j}({}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j + \Omega_j) \end{aligned}$$

and the products of inertia

$$\begin{aligned} \tilde{J}_{12j} =& J_{11j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j - {}^i\dot{\gamma}_j \mathrm{s}\theta_j) - J_{13j}({}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j + \Omega_j) \\ &+ J_{22j}({}^i\dot{\gamma}_j \mathrm{s}\theta_j - {}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j) \\ &+ J_{23j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ \tilde{J}_{13j} =& -J_{11j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ &+ J_{23j}({}^i\dot{\gamma}_j \mathrm{s}\theta_j - {}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j) \\ &+ J_{33j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ &+ J_{12j}({}^o\dot{\gamma}_j \sin{}^i\gamma_j + \Omega_j) \\ \tilde{J}_{23j} =& J_{13j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{c}\theta_j - {}^i\dot{\gamma}_j \mathrm{s}\theta_j) \\ &- J_{12j}({}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j \mathrm{s}\theta_j + {}^i\dot{\gamma}_j \mathrm{c}\theta_j) \\ &+ J_{22j}({}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j + \Omega_j) - J_{33j}({}^o\dot{\gamma}_j \mathrm{s}^i\gamma_j + \Omega_j) \end{aligned}$$

Furthermore, by assuming $J_{12j} = J_{23j} = 0$, $J_{22j} = J_{33j}$, $J_{13} = 0$ and $\theta_j = 0$ (since the wheel frame does not rotate), the equation above simplifies to

$$[\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' = {}^{\mathcal{W}_j}\begin{bmatrix} 0 & J_{aj}{}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j & -J_{aj}{}^i\dot{\gamma}_j \\ J_{aj}{}^o\dot{\gamma}_j \mathrm{c}^i\gamma_j & 0 & 0 \\ -J_{aj}{}^i\dot{\gamma}_j & 0 & 0 \end{bmatrix} \tag{65}$$

where

$$J_{aj} = J_{11j} - J_{22j} \tag{66}$$

The second inertial derivative of $\boldsymbol{r}_{W_{cj}/B}$ is needed. Define the body frame derivative and first inertial derivative of $\boldsymbol{r}_{G^i_{cj}/B}$, noting that W_j is fixed with respect to point B.

$$\boldsymbol{r}_{W_{cj}/B} = \boldsymbol{r}_{G^i_{cj}/G^i_j} + \boldsymbol{r}_{G^i_j/G^o_j} + \boldsymbol{r}_{G^o_j/B} \tag{67}$$

$$\dot{\boldsymbol{r}}_{W_{cj}/B} = \boldsymbol{r}'_{W_{cj}/B} + \boldsymbol{\omega} \times \boldsymbol{r}_{W_{cj}/B} \tag{68}$$

The second inertial derivative of $\boldsymbol{r}_{G^i_{cj}/B}$ is given by

$$\begin{aligned} \ddot{\boldsymbol{r}}_{W_{cj}/B} =& \boldsymbol{r}''_{W_{cj}/B} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{W_{cj}/B} + \boldsymbol{\omega} \times \boldsymbol{r}'_{W_{cj}/B} \\ &+ \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{W_{cj}/B} \\ =& \boldsymbol{r}''_{W_{cj}/B} + [\tilde{\boldsymbol{r}}_{W_{cj}/B}]^T \dot{\boldsymbol{\omega}} + 2 \times [\tilde{\boldsymbol{r}}'_{W_{cj}/B}]^T \boldsymbol{\omega} \\ &+ [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{W_{cj}/B}. \end{aligned} \tag{69}$$

The second body frame derivative of $\boldsymbol{r}_{W_{cj}/B}$ is defined in Eq. (14). Substituting Eq. (69) into Eq. (62) results in

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{W}_j,B} =& [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j \hat{\boldsymbol{s}}_{xj} \\ &+ {}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} \\ &+ {}^i\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} + m_{\mathrm{W}_j}[\tilde{\boldsymbol{r}}_{W_{cj}/B}] \\ &\Big[\boldsymbol{r}''_{W_{cj}/B} + [\tilde{\boldsymbol{r}}_{W_{cj}/B}]^T \boldsymbol{\omega} + 2 \times [\dot{\tilde{}}'_{W_{cj}/B}]^T \boldsymbol{\varpi} \\ &+ [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{W_{cj}/B}\Big] \end{aligned} \tag{70}$$

The parallel axis theorem relating the wheel inertia about point B to the wheel inertia about point W_{cj} is given by

$$[\boldsymbol{I}_{\mathrm{W}_j,B}] = [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}] + m_{W_j}[\tilde{\boldsymbol{r}}_{W_{cj}/B}][\tilde{\boldsymbol{r}}_{W_{cj}/B}]^T \tag{71}$$

Using Eq. (70), Eq. (71) simplifies to

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{W}_j,B} =& [\boldsymbol{I}_{\mathrm{W}_j,B}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{W_j,W_{cj}}]({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} \\ &+ \dot{\Omega}_j \hat{\boldsymbol{s}}_{xj} + {}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} \\ &+ {}^i\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,B}]\boldsymbol{\omega} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{B}} + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \\ &+ m_{\mathrm{W}_j}[\tilde{\boldsymbol{r}}_{W_{cj}/B}]\Big[\boldsymbol{r}''_{W_{cj}/B} + 2[\tilde{\boldsymbol{r}}'_{W_{cj}/B}]^T \boldsymbol{\omega}\Big] \end{aligned} \tag{72}$$

Employing the body frame derivative of the parallel axis theorem.

$$\begin{aligned} [\boldsymbol{I}_{\mathrm{W}_j,B}]' =& [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' + m_{W_j}[\tilde{\boldsymbol{r}}'_{W_{cj}/B}][\tilde{\boldsymbol{r}}_{W_{cj}/B}]^T \\ &+ m_{W_j}[\tilde{\boldsymbol{r}}_{W_{cj}/B}][\tilde{\boldsymbol{r}}'_{W_{cj}/B}]^T \end{aligned} \tag{73}$$

Eq. (72) is further simplified using Eq. (73) to give the outer gimbal angular momentum derivative.

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{W}_j,B} =& [\boldsymbol{I}_{\mathrm{W}_j,B}]\dot{\boldsymbol{\omega}} + [\boldsymbol{I}_{\mathrm{W}_j,B}]'\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,B}]\boldsymbol{\omega} \\ &+ [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]({}^o\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j \hat{\boldsymbol{s}}_{xj} \\ &+ {}^o\dot{\gamma}_j {}^i\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} \\ &+ {}^i\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{B}} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{B}} + m_{\mathrm{W}_j}[\tilde{\boldsymbol{r}}_{W_{cj}/B}]\boldsymbol{r}''_{W_{cj}/B} \\ &+ m_{\mathrm{W}_j}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{W_{cj}/B}]\boldsymbol{r}'_{W_{cj}/B}. \end{aligned} \tag{74}$$

We may now formulate the rotational equation of motion. Euler's equation is rearranged as

$$m_{sc}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{H}}_{\text{B},B} + \sum_{j=1}^{N}(\dot{\boldsymbol{H}}_{G_j^o,B} + \dot{\boldsymbol{H}}_{G_j^i,B} + \dot{\boldsymbol{H}}_{W_j,B}) = \boldsymbol{L}_B \quad (75)$$

The rotational equations of motion are formulated by substituting Eqs. (30), (43), (58) and (74) into Eq. (75) and the total spacecraft inertia about point B is given by

$$[\boldsymbol{I}_{\text{sc},B}] = [\boldsymbol{I}_{\text{B},B}] + \sum_{j=1}^{N}\Big([\boldsymbol{I}_{\text{G}_j^o,B}] + [\boldsymbol{I}_{\text{G}_j^i,B}] + [\boldsymbol{I}_{\text{W}_j,B}]\Big) \quad (76)$$

This equation represents the general form of a spacecraft with balance/imbalance momentum exchange devices such as DGVSCMG, DGCMG, VSCMG, SGCMG and RW.

3.3 Motor Torque Equations

3.3.1 Outer Gimbal Torque Equation

The outer gimbal torque equation is used to relate the body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and the outer gimbal derivative ${}^o\ddot{\gamma}_j$. The outer gimbal moter torque u_{zj} is the $\hat{\boldsymbol{s}}_{zj}$ component of gimbal torque about point G_j^o. The torque acting on a DGVSCMG at the joint between the motor and the inner/outer gimbal assembly is given by

$$\boldsymbol{L}_{G_j^o} = {}^{\mathcal{G}_j^o}\begin{bmatrix} {}^o\tau_{xj} \\ {}^o\tau_{yj} \\ u_{zj} \end{bmatrix} \quad (77)$$

The transverse toorque acting on the gimbal ${}^o\tau_{xj}$ and ${}^o\tau_{yj}$ are structural torques and do not contribute to the equation. Torque about point G_j^o is related to torque about the DGVSCMG center of mass D_{cj} using the following equation.

$$\boldsymbol{L}_{G_j^o} = \boldsymbol{L}_{D_j} + \boldsymbol{r}_{D_{cj}/G_j^o} \times m_{D_{cj}}\ddot{\boldsymbol{r}}_{D_{cj}/N} \quad (78)$$

Euler's equation applies as follows.

$$\boldsymbol{L}_{D_j} = \dot{\boldsymbol{H}}_{\text{G}_j^o,D_{cj}} + \dot{\boldsymbol{H}}_{\text{G}_j^i,D_{cj}} + \dot{\boldsymbol{H}}_{\text{W}_j,D_{cj}} \quad (79)$$

The outer gimbal motor torque is the $\hat{\boldsymbol{s}}_{zj}$ component of the right-hand side of Eq. (78). This is found in a frame independent format as

$$u_{zj} = \hat{\boldsymbol{s}}_{zj}^T\boldsymbol{L}_{G_j^o} = \hat{\boldsymbol{s}}_{zj}^T(\dot{\boldsymbol{H}}_{\text{G}_j^o,D_{cj}} + \dot{\boldsymbol{H}}_{\text{G}_j^i,D_{cj}} + \dot{\boldsymbol{H}}_{\text{W}_j,D_{cj}} + \boldsymbol{r}_{D_{cj}/G_j^o} \times m_{D_{cj}}\ddot{\boldsymbol{r}}_{D_{cj}/N}) \quad (80)$$

where the outer/inner gimbal and wheel angular momentum derivatives about point D_{cj} are related to point G_{cj}^o using the following equation.

$$\dot{\boldsymbol{H}}_{\text{G}_j^o,D_{cj}} = \dot{\boldsymbol{H}}_{\text{G}_j^o,G_{cj}^o} + m_{G_j^o}\boldsymbol{r}_{G_{cj}^o/D_{cj}} \times \ddot{\boldsymbol{r}}_{G_{cj}^o/D_{cj}} \quad (81)$$

$$\dot{\boldsymbol{H}}_{\text{G}_j^i,D_{cj}} = \dot{\boldsymbol{H}}_{\text{G}_j^i,G_{cj}^i} + m_{G_j^i}\boldsymbol{r}_{G_{cj}^i/D_{cj}} \times \ddot{\boldsymbol{r}}_{G_{cj}^i/D_{cj}} \quad (82)$$

$$\dot{\boldsymbol{H}}_{\text{W}_j,D_{cj}} = \dot{\boldsymbol{H}}_{\text{W}_j,W_{cj}} + m_{W_j}\boldsymbol{r}_{W_{cj}/D_{cj}} \times \ddot{\boldsymbol{r}}_{W_{cj}/D_{cj}} \quad (83)$$

The inertial derivatives of the wheel and inner/outer gimbal angular momentum about their respective centers of mass were found in the previous section and are reprinted here for the reader's convenience.

$$\dot{\boldsymbol{H}}_{\text{G}_j^o,G_{cj}^o} = [\boldsymbol{I}_{\text{G}_j^o,G_{cj}^o}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj}) + [\boldsymbol{I}_{\text{G}_j^o,G_{cj}^o}]'\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\text{G}_j^o,G_{cj}^o}]\boldsymbol{\omega}_{\mathcal{G}_j^o/\mathcal{N}} \quad (84)$$

$$\dot{\boldsymbol{H}}_{\text{G}_j^i,G_{cj}^i} = [\boldsymbol{I}_{\text{G}_j^i,G_{cj}^i}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j\hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj}) + [\boldsymbol{I}_{\text{G}_j^i,G_{cj}^i}]'\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\text{G}_j^i,G_{cj}^i}]\boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \quad (85)$$

$$\dot{\boldsymbol{H}}_{\text{W}_j,W_{cj}} = [\boldsymbol{I}_{\text{W}_j,W_{cj}}](\dot{\boldsymbol{\omega}} + {}^o\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^i\ddot{\gamma}_j\hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j\hat{\boldsymbol{s}}_{xj} + {}^o\dot{\gamma}_j{}^i\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^o\dot{\gamma}_j\Omega_j\hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} + {}^i\dot{\gamma}_j\Omega_j\hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\text{W}_j,W_{cj}}]'\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\text{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \quad (86)$$

Next the $\ddot{\boldsymbol{r}}_{G_{cj}^o/D_{cj}}$ term in Eq. (81) is defined as

$$\ddot{\boldsymbol{r}}_{G_{cj}^o/D_{cj}} = \boldsymbol{r}''_{G_{cj}^o/D_{cj}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{cj}^o/D_{cj}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{G_{cj}^o/D_{cj}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{G_{cj}^o/D_{cj}}) \quad (87)$$

The vector $\ddot{\boldsymbol{r}}_{G_{cj}^i/D_{cj}}$ in Eq. (82) is expressed as

$$\ddot{\boldsymbol{r}}_{G_{cj}^i/D_{cj}} = \boldsymbol{r}''_{G_{cj}^i/D_{cj}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{cj}^i/D_{cj}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{G_{cj}^i/D_{cj}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{G_{cj}^i/D_{cj}}) \quad (88)$$

The DGVSCMG center of mass location with respect to point G_j^o and its body frame derivatives are given by

$$\boldsymbol{r}_{D_{cj}/G_{cj}^o} = \frac{1}{m_{D_j}}\Big(m_{G_j^o}\boldsymbol{r}_{G_{cj}^o/G_j^o} + m_{G_j^i}\boldsymbol{r}_{G_{cj}^i/G_j^o} + m_{W_j}\boldsymbol{r}_{W_{cj}/G_j^o}\Big) \quad (89)$$

$$\boldsymbol{r}'_{D_{cj}/G_{cj}^o} = {}^o\rho_{G_j^o}\boldsymbol{r}'_{G_{cj}^o/G_j^o} + {}^i\rho_{G_j^i}\boldsymbol{r}'_{G_{cj}^i/G_j^i} + {}^o\rho_{W_j}\boldsymbol{r}'_{W_{cj}/W_j} \quad (90)$$

$$\boldsymbol{r}''_{D_{cj}/G_{cj}^o} = {}^o\rho_{G_j^o}\boldsymbol{r}''_{G_{cj}^o/G_j^o} + {}^i\rho_{G_j^i}\boldsymbol{r}''_{G_{cj}^i/G_j^i} + {}^o\rho_{W_j}\boldsymbol{r}''_{W_{cj}/W_j} \quad (91)$$

where the mass ratios are abbreviated as

$$ {}^{o}\rho_{G_j^o} = \frac{m_{G_j^o}}{m_{G_j^o} + m_{G_j^i} + m_{W_j}} \tag{92} $$

$$ {}^{i}\rho_{G_j^i} = \frac{m_{G_j^i}}{m_{G_j^o} + m_{G_j^i} + m_{W_j}} \tag{93} $$

$$ {}^{w}\rho_{W_j} = \frac{m_{W_j}}{m_{G_j^o} + m_{G_j^i} + m_{W_j}} \tag{94} $$

Next the term $\ddot{\boldsymbol{r}}_{D_{cj}/N}$ is evaluated

$$ \ddot{\boldsymbol{r}}_{D_{cj}/N} = \ddot{\boldsymbol{r}}_{D_{cj}/B} + \ddot{\boldsymbol{r}}_{B/N} \tag{95} $$

Finally the second inertial derivative of $\boldsymbol{r}_{D_{cj}/B}$ is found noting that $\boldsymbol{r}'_{D_{cj}/G_j^o} = \boldsymbol{r}'_{D_{cj}/B}$ and $\boldsymbol{r}''_{D_{cj}/G_j^o} = \boldsymbol{r}''_{D_{cj}/B}$.

$$ \dot{\boldsymbol{r}}_{D_{cj}/B} = \boldsymbol{r}'_{D_{cj}/B} + \boldsymbol{\omega} \times \boldsymbol{r}_{D_{cj}/B} \tag{96} $$

$$ \ddot{\boldsymbol{r}}_{D_{cj}/B} = \boldsymbol{r}''_{D_{cj}/B} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{D_{cj}/B} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{D_{cj}/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{D_{cj}/B}). \tag{97} $$

3.3.2 Inner Gimbal Torque Equation

The inner gimbal torque equation is used to relate body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and the inner gimbal derivative ${}^{i}\ddot{\gamma}_j$. The inner gimbal moter torque u_{yj} is the $\hat{\boldsymbol{s}}_{yj}$ component of gimbal torque about point G_j^i. The torque acting on a DGVSCMG at the joint between the motor and the inner/outer gimbal assembly is given by

$$ \boldsymbol{L}_{G_j^i} = {}^{\mathcal{G}_j^i}\begin{bmatrix} {}^{i}\tau_{xj} \\ u_{yj} \\ {}^{i}\tau_{zj} \end{bmatrix} \tag{98} $$

The transverse toorque acting on the gimbal ${}^{i}\tau_{xj}$ and ${}^{i}\tau_{zj}$ are structural torques and do not contribute to the equation. Torque about point G_j^i is related to torque about the DGVSCMG center of mass D_{cj} using the following equation.

$$ \boldsymbol{L}_{G_j^i} = \boldsymbol{L}_{V_j} + \boldsymbol{r}_{V_{cj}/G_j^i} \times m_{V_{cj}} \ddot{\boldsymbol{r}}_{D_{cj}/N} \tag{99} $$

Euler's equation applies as follows.

$$ \boldsymbol{L}_{V_j} = \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,V_{cj}} + \dot{\boldsymbol{H}}_{\mathrm{W}_j,V_{cj}} \tag{100} $$

The inner gimbal motor torque is the ${}^{i}\hat{\boldsymbol{s}}_{yj}$ component of the right-hand side of Eq. (99). This is found in a frame independent format as

$$ u_{yj} = \hat{\boldsymbol{s}}_{yj}^T \boldsymbol{L}_{G_j^i} = \hat{\boldsymbol{s}}_{yj}^T \Big(\dot{\boldsymbol{H}}_{\mathrm{G}_j^i,V_{cj}} + \dot{\boldsymbol{H}}_{\mathrm{W}_j,V_{cj}} + \boldsymbol{r}_{V_{cj}/G_j^i} \times m_{V_{cj}} \ddot{\boldsymbol{r}}_{V_{cj}/N} \Big) \tag{101} $$

where the outer/inner gimbal and wheel angular momentum derivatives about point V_{cj} are related to point G_{cj}^i using the following equation.

$$ \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,V_{cj}} = \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,G_{cj}^i} + m_{G_j^i} \boldsymbol{r}_{G_{cj}^i/V_{cj}} \times \ddot{\boldsymbol{r}}_{G_{cj}^i/V_{cj}} \tag{102} $$

$$ \dot{\boldsymbol{H}}_{\mathrm{W}_j,V_{cj}} = \dot{\boldsymbol{H}}_{\mathrm{W}_j,W_{cj}} + m_{W_j} \boldsymbol{r}_{W_{cj}/V_{cj}} \times \ddot{\boldsymbol{r}}_{W_{cj}/V_{cj}} \tag{103} $$

The inertial derivatives of the wheel and inner/outer gimbal angular momentum about their respective centers of mass were found in the previous section and are reprinted here for the reader's convenience.

$$ \dot{\boldsymbol{H}}_{\mathrm{G}_j^i,G_{cj}^i} = [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}](\dot{\boldsymbol{\omega}} + {}^{o}\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^{i}\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + {}^{o}\dot{\gamma}_j {}^{i}\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj}) + [\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}]' \boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{G}_j^i,G_{cj}^i}] \boldsymbol{\omega}_{\mathcal{G}_j^i/\mathcal{N}} \tag{104} $$

$$ \dot{\boldsymbol{H}}_{\mathrm{W}_j,W_{cj}} = [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}](\dot{\boldsymbol{\omega}} + {}^{o}\ddot{\gamma}_j \hat{\boldsymbol{s}}_{zj} + {}^{i}\ddot{\gamma}_j \hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j \hat{\boldsymbol{s}}_{xj} + {}^{o}\dot{\gamma}_j {}^{i}\dot{\gamma}_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^{o}\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} + {}^{i}\dot{\gamma}_j \Omega_j \hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]' \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}] \boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \tag{105} $$

The acceleration vector $\ddot{\boldsymbol{r}}_{G_{cj}^i/V_{cj}}$ is expressed as

$$ \ddot{\boldsymbol{r}}_{G_{cj}^i/V_{cj}} = \boldsymbol{r}''_{G_{cj}^i/V_{cj}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{cj}^i/V_{cj}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{G_{cj}^i/V_{cj}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{G_{cj}^i/V_{cj}}) \tag{106} $$

while the acceleration $\ddot{\boldsymbol{r}}_{W_{cj}/V_{cj}}$ is defined as

$$ \ddot{\boldsymbol{r}}_{W_{cj}/V_{cj}} = \boldsymbol{r}''_{W_{cj}/V_{cj}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{W_{cj}/V_{cj}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{W_{cj}/V_{cj}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{W_{cj}/V_{cj}}) \tag{107} $$

The DGVSCMG center of mass location with respect to point G_j^i and its body frame derivatives are given by

$$ \boldsymbol{r}_{V_{cj}/G_{cj}^i} = \frac{1}{m_{V_j}} (m_{G_j^i} \boldsymbol{r}_{G_{cj}^i/G_j^o} + m_{W_j} \boldsymbol{r}_{W_{cj}/G_j^i}) \tag{108} $$

$$ \boldsymbol{r}'_{V_{cj}/G_{cj}^i} = {}^{i}\rho_{G_j^i} \boldsymbol{r}'_{G_{cj}^i/G_j^i} + {}^{i}\rho_{W_j} \boldsymbol{r}'_{W_{cj}/G_j^i} \tag{109} $$

$$ \boldsymbol{r}''_{V_{cj}/G_{cj}^i} = {}^{i}\rho_{G_j^i} \boldsymbol{r}''_{G_{cj}^i/G_j^i} + {}^{i}\rho_{W_j} \boldsymbol{r}''_{W_{cj}/G_j^i} \tag{110} $$

where

$$ {}^{i}\rho_{G_j^i} = \frac{m_{G_j^i}}{m_{G_j^i} + m_{W_j}} \tag{111} $$

$$ {}^{i}\rho_{W_j} = \frac{m_{W_j}}{m_{G_j^i} + m_{W_j}} \tag{112} $$

Evaluating $\ddot{\boldsymbol{r}}_{V_{cj}/N}$ yields

$$ \ddot{\boldsymbol{r}}_{V_{cj}/N} = \ddot{\boldsymbol{r}}_{V_{cj}/B} + \ddot{\boldsymbol{r}}_{B/N} \tag{113} $$

Finally the second inertial derivative of $\boldsymbol{r}_{V_{cj}/B}$ is found noting that $\boldsymbol{r}'_{V_{cj}/G_j^i} = \boldsymbol{r}'_{V_{cj}/B}$ and $\boldsymbol{r}''_{V_{cj}/G_j^i} = \boldsymbol{r}''_{V_{cj}/B}$.

$$ \dot{\boldsymbol{r}}_{V_{cj}/B} = \boldsymbol{r}'_{V_{cj}/B} + \boldsymbol{\omega} \times \boldsymbol{r}_{V_{cj}/B} \tag{114} $$

$$ \ddot{\boldsymbol{r}}_{V_{cj}/B} = \boldsymbol{r}''_{V_{cj}/B} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{V_{cj}/B} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{V_{cj}/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{V_{cj}/B}) \tag{115} $$

3.3.3 Wheel Torque Equation

The wheel torque equation is used to relate body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and the wheel speed derivative $\dot{\Omega}_j$. The wheel moter torque u_{xj} is the $\hat{\boldsymbol{s}}_{xj}$ component of wheel torque about point W_j. The torque acting on a RW at the joint between the RW motor and the RW rotor is given by

$$\boldsymbol{L}_{W_j} = {}^{\mathcal{W}_j}\begin{bmatrix} u_{xj} \\ {}^{w}\tau_{yj} \\ {}^{w}\tau_{zj} \end{bmatrix} \tag{116}$$

The transverse toorque acting on the gimbal ${}^{w}\tau_{yj}$ and ${}^{w}\tau_{zj}$ are structural torques and do not contribute to the equation. Torque about point W_j is related to torque about the center of mass W_{cj} using the following equation.

$$\boldsymbol{L}_{W_j} = \boldsymbol{L}_{W_{cj}} + \boldsymbol{r}_{W_{cj}/W_j} \times m_{W_j}\ddot{\boldsymbol{r}}_{W_{cj}/N} \tag{117}$$

Euler's equation applies as follows.

$$\boldsymbol{L}_{W_{cj}} = \dot{\boldsymbol{H}}_{\mathrm{W}_j,W_{cj}} \tag{118}$$

The motor torque is the ${}^{i}\hat{\boldsymbol{s}}_{xj}$ component of the right-hand side of Eq. (117). This is found in a frame independent format as

$$\begin{aligned} u_{xj} = \hat{\boldsymbol{s}}_{xj}^T\boldsymbol{L}_{W_j} = \hat{\boldsymbol{s}}_{xj}^T(&\dot{\boldsymbol{H}}_{\mathrm{W}_j,W_{cj}} \\ &+\boldsymbol{r}_{W_{cj}/W_j} \times m_{W_j}\ddot{\boldsymbol{r}}_{W_{cj}/N}) \end{aligned} \tag{119}$$

The inertial derivatives of the wheel and inner/outer gimbal angular momentum about their respective centers of mass were found in the previous section and are reprinted here for the reader's convenience.

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{W}_j,W_{cj}} =& [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}](\dot{\boldsymbol{\omega}} + {}^{o}\ddot{\gamma}_j\hat{\boldsymbol{s}}_{zj} + {}^{i}\ddot{\gamma}_j\hat{\boldsymbol{s}}_{yj} + \dot{\Omega}_j\hat{\boldsymbol{s}}_{xj} \\ &+ {}^{o}\dot{\gamma}_j{}^{i}\dot{\gamma}_j\hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{yj} + {}^{o}\dot{\gamma}_j\Omega_j\hat{\boldsymbol{s}}_{zj} \times \hat{\boldsymbol{s}}_{xj} \\ &+ {}^{i}\dot{\gamma}_j\Omega_j\hat{\boldsymbol{s}}_{yj} \times \hat{\boldsymbol{s}}_{xj}) + [\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]'\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \\ &+ [\tilde{\boldsymbol{\omega}}][\boldsymbol{I}_{\mathrm{W}_j,W_{cj}}]\boldsymbol{\omega}_{\mathcal{W}_j/\mathcal{N}} \end{aligned} \tag{120}$$

The acceleration vector $\ddot{\boldsymbol{r}}_{W_{cj}/N}$ is defined as

$$\begin{aligned} \ddot{\boldsymbol{r}}_{W_{cj}/N} = \boldsymbol{r}''_{W_{cj}/B} + [\tilde{\boldsymbol{r}}_{W_{cj}/B}]^T\dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{W_{cj}/B}]^T\boldsymbol{\omega} \\ + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{W_{cj}/B} + \ddot{\boldsymbol{r}}_{B/N} \end{aligned} \tag{121}$$

4. Numerical Simulations

The simulation results show the angular velocity of the spacecraft as in Fig. (3) and the inertial position of the spacecraft as in Fig. (4). The fully-coupled model parameters in Table 1 closely parallel those used in Refs. 9 and 11. The scenario used to demonstrate the fully-coupled imbalanced DGVSCMG EOM involves a rigid spacecraft hub and N = 1 DGVSCMG. The lumped manufacturer imbalance parameters are related to the parameters used within this derivation using the imbalance parameter adaptation formulation given in Ref. 9. The static imbalance is given by

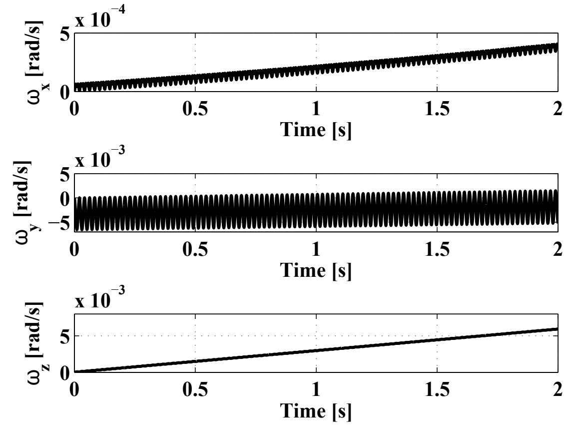

Fig. 3: Angular velocity of the spacecraft.

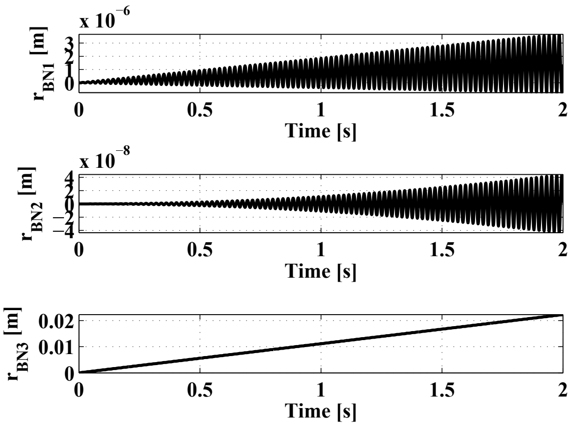

Fig. 4: Inertial position of the spacecraft.

$$d_i = \frac{U_{sj}}{m_{rwj}}, \tag{122}$$

where U_{sj} is the wheel static imbalance parameter. The dynamic imbalance parameter is given by the following constraint:

$$U_{dj} = \sqrt{J_{13j}^2 + J_{12j}^2}, \tag{123}$$

Some combination of J_{12j} and J_{13j} must be selected for each wheel such that Eq. (123) is satisfied. Since the dynamic imbalanced parameter is determined by the arbitrary vector, the following definitions are chosen

$$J_{13j} = U_{dj} \tag{124}$$

$$J_{12j} = J_{23j} = 0. \tag{125}$$

From Figs. (3) and (4), the magnitude of the imbalance vibration effect on the spacecraft velocity or the spacecraft position can be estimated.

Table 1: Simulation parameters.

Parameters	Value	Units
N	1	–
m_{sc}	862	kg
m_B	810	kg
m_w	4	kg
m_{G^i}	24	kg
m_{G^o}	24	kg
$[\boldsymbol{I}_{\text{hub},B_c}]$	${}^{\mathcal{B}}\begin{bmatrix} 900 & 4.15 & 2.93 \\ 4.15 & 800 & 2.75 \\ 2.93 & 2.75 & 600 \end{bmatrix}$	$\text{kg}\cdot\text{m}^2$
$\boldsymbol{r}_{B_c/B}$	${}^{\mathcal{B}}\begin{bmatrix}-0.02 & 0.01 & 10\end{bmatrix}^T$	cm
U_s	32	$\text{g}\cdot\text{cm}$
U_d	15.4	$\text{g}\cdot\text{m}^2$
d	8.0	mm
$[\boldsymbol{I}_{\text{W},W_c}]$	${}^{\mathcal{G}_i}\begin{bmatrix} 0.2 & 0 & 0.0154 \\ 0 & 0.1 & 0 \\ 0.0154 & 0 & 0.1 \end{bmatrix}$	$\text{kg}\cdot\text{m}^2$
$[\boldsymbol{I}_{\text{G}^i,G_c^i}]$	${}^{\mathcal{G}_i}\begin{bmatrix} 9 & 0.81 & 0.24 \\ 0.81 & 11 & 0.93 \\ 0.24 & 0.93 & 5 \end{bmatrix}$	$\text{kg}\cdot\text{m}^2$
$[\boldsymbol{I}_{\text{G}^o,G_c^o}]$	${}^{\mathcal{G}_o}\begin{bmatrix} 9 & 0.81 & 0.24 \\ 0.81 & 11 & 0.93 \\ 0.24 & 0.93 & 5 \end{bmatrix}$	$\text{kg}\cdot\text{m}^2$
$\boldsymbol{r}_{G^o/B}$	${}^{\mathcal{B}}\begin{bmatrix}30 & 0 & 0\end{bmatrix}^T$	cm
$\boldsymbol{r}_{B/N}$	${}^{\mathcal{N}}\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$	m
$\boldsymbol{v}_{B/N}$	${}^{\mathcal{N}}\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$	m/s
$\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}}$	$\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$	–
$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$	${}^{\mathcal{B}}\begin{bmatrix}4.85 & 0.57 & 0\end{bmatrix}^T$	deg/s
Ω	500	rpm
θ	0	deg
${}^i\dot{\gamma}$	0	deg/s
${}^i\gamma$	0	deg
${}^o\dot{\gamma}$	0	deg/s
${}^o\gamma$	0	deg
u_x	250	$\text{mN}\cdot\text{m}$
u_y	100	$\text{mN}\cdot\text{m}$
u_z	100	$\text{mN}\cdot\text{m}$

5. Conclusion/Future Work

Most previous work related to modeling static and dynamic imbalances of momentum exchange device (MED) models the effect as an external force and torque on the spacecraft. In reality, this effect is an internal force and torque on the spacecraft and thus requires a different formulation. The work presented in this paper develops the general fully-coupled model of double-gimbal variable-speed control moment gyros (DGVSCMGs) imbalances. This developed model is generalized description of a fully-coupled dynamical jitter model for a spacecraft with MEDs since the equation of motion (EOM) of DGVSCMGs includes the EOM of all MEDs such as reaction wheel (RW), single-gimbal CMG or VSCMG.

As future works, the energy or energy rate of a fully-coupled model of multiple-DGVSCMGs will be investigated. By using this fully-couple model, a high-precision attitude control or fault detection method will be considered.

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REFERENCES

[1] D. Stevenson and H. Schaub, Nonlinear Control Analysis of a Double-Gimbal Variable-Speed Control Moment Gyroscope, *Journal of Guidance, Control, and Dynamics,* Vol. 35, No. 3, May-June (2012) 787-793.

[2] H. Zhang and J. Fang, Robust Backstepping Control for Agile Satellite Using Double-Gimbal Variable-Speed Control Moment Gyroscope, *Journal of Guidance, Control, and Dynamics*, Vol. 36, No. 5, September–October (2013) 1356-1363.

[3] I. Jikuya, K. Fujii and K. Yamada, Attitude Maneuver of Spacecraft with a Variable-Speed Double-Gimbal Ccontrol Mmoment Gyro, *Advances in Space Research,* Vol. 58, No. 7, October (2016) 1303-1317.

[4] K. Liu, P. Maghami and C. Blaurock, Reaction Wheel Disturbance Modeling, Jitter Analysis, and Validation Tests for Solar Dynamics Observatory, (2008).

[5] S. Miller, P. Kirchman and J. Sudey, Reaction Wheel Operational Impacts On the GOES-N Jitter Environment, *AIAA Guidance, Navigation and Control Conference and Exhibit, AIAA2007-6736* (2007).

[6] D. Kim, Micro-Vibration Model and Parameter Estimation Method of a Reaction Wheel Assembly, *Journal of Sound and Vibration*, Vol. 333, No. 18 (2014) 4214-4231.

[7] K. C. Liu, P. Maghami, and C. Blaurock, "Reaction Wheel Disturbance Modeling, Jitter Analysis, and Validation Tests for Solar Dynamics Observatory", *AIAA Guidance, Navigation and Control Conference and Exhibit, AIAA2008-7232* (2008).

[8] K. J. Pendergast and C. J. Schauwecker, "Use of a Passive Reaction Wheel Jitter Isolation System to Meet the Advanced X-ray Astrophysics Facility Imaging Performance Requirements", *NASA/CR-1998-207926* (1998).

[9] J. Alcorn, C. Allard and H. Schaub, Fully-Coupled Dynamical Jitter Modeling of a Rigid Spacecraft With Imbalanced Reaction Wheels, *AIAA SPACE 2016*, AIAA 2016-5686, Long Beach, CA, September (2016).

[10] Y. Zhang and J. Zhang, Disturbance Characteristics Analysis of CMG Due to Imbalances and Installation Errors, *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 50, No. 2 (2014) 1017-1026.

[11] J. Alcorn, C. Allard and H. Schaub, Fully-Coupled Dynamical Jitter Modeling of Variable-Speed Control Moment Gyroscopes, *AAS/AIAA Astrodynamics Specialists Conference 2017*, AAS 17-730, Stevenson, WA, August (2017).

[12] D. Li, X. Chen and B. Wu, Analysis of Reaction-Wheels Imbalance Torque Effects On Satellite Attitude Control System, Chinese Control and Decision Conference 2016, (2016) 3722-3725.