(Preprint) AAS 24-170

SPACECRAFT DYNAMICS CONTAINING MOTION PLATFORMS WITH DYNAMIC SUB-COMPONENTS

Leah Kiner* and Hanspeter Schaub†

The ability to model, simulate, and analyze complex spacecraft designs is crucial to verify mission requirements and ensure the long-term success of space missions. Previous work used the Backsubstitution Method to formulate the dynamics for N chained translational or rotational components attached to a central rigid hub. This work develops the new ability to simulate select branching of spacecraft components using the Backsubstitution Method, where ultimately N force or torque-actuated sub-components can be directly attached to a hub-connected prescribed motion platform. The dynamics for the general system consisting of a hub-connected prescribed motion body with a single attached dynamic sub-component are derived and implemented into the Backsubstitution Method for future integration into a spacecraft simulation software. The modularity of the derived equations enables M of the described multi-body actuator components to be connected to the central rigid hub.

INTRODUCTION

Spacecraft design concepts have seen a drastic evolution since the late 1950s. Advancements in technology coupled with increasing ambition to explore the farthest edges of our solar system have contributed to the swift progression from modest, single-bodied designs containing only external radio transmission antennas to immense, multi-body structures with an abundance of actuated components onboard. For example, deep space missions such as the Lucy mission to the Trojan asteroids, the EMA mission to the main asteroid belt, and the Dart binary asteroid impact mission have required solar array and thruster design advancements in order to meet mission power, momentum storage, and thrust-vector alignment requirements. To meet power generation needs, the Lucy and EMA mission designs feature a central spacecraft hub with two large symmetrically-attached circular flexible-substrate solar arrays. These arrays articulate using stepper motors to track the Sun and are the first of their kind to deploy using a motor-driven lanyard.[1–3] Similarly, the Dart spacecraft uses two articulated rectangular flexible-substrate arrays to track the Sun. The arrays demonstrated a new roll-out technology method to deploy the large 8 meter long arrays.[4] Further, to meet thrust-vector alignment requirements for deep space missions such as EMA,[5] Deep Space 1,[6] Dawn,[7] and Psyche,[8] spacecraft ionic thruster designs have advanced from hub-fixed configurations to dual-axis gimballed configurations.[5,9–11] Moreover, the interest to establish the presence of humans in space drove advancements in space orbiters such as the Space Shuttle and the ISS,

*Graduate Research Assistant, Ann and H.J. Smead Department of Aerospace Engineering Sciences, University of Colorado Boulder, Colorado Center For Astrodynamics Research, Boulder, CO, 80303 USA. leah.kiner@colorado.edu
†Professor and Department Chair, Schaden Leadership Chair, Ann and H.J. Smead Department of Aerospace Engineering Sciences, University of Colorado, Boulder, 431 UCB, Colorado Center for Astrodynamics Research, Boulder, CO, 80309. AAS Fellow, AIAA Fellow.

where multi-link robotic arms such as the Canadarm[12–14] were developed to aid in complex orbital servicing and docking operations.

There are an abundance of existing methods that could be chosen to derive the dynamics of these complex multi-body systems. Although commercial software packages such as COMSOL,[15] Adams,[16] and MathWork's Simscape Multibody[17] are available to compute and provide the equations of motion for intricate multi-body systems, it is especially important to choose a method that is both general and computationally efficient so that a wide range of spacecraft configurations can be modeled and simulated rapidly in software. These software packages are unable to provide generalized dynamics formulations. Using Newtonian and Eularian mechanics, this work leverages the Backsubstitution Method[18–20] to generally and modularly develop the spacecraft system equations of motion so that they may be readily implemented into a spacecraft simulation software. Relying on a hub-centric spacecraft design, this method achieves greater computational efficiency compared to other software tools by analytically back-solving all component coupling to the central rigid hub in order to avoid inversion of the entire system mass matrix.

The Backsubstitution Method has been demonstrated for a wide variety of hub-centric spacecraft configurations including single and N-hinged solar panels,[21,22] reaction wheels,[23] control moment gyroscopes,[24] chained rotational bodies,[25,26] chained translational bodies,[27] and general prescribed motion components.[3,11,28,29] While these formulations are useful to simulate chains of hub-connected spacecraft components such as robotic arms, they are limited in that no formulations have enabled branching of elements relative to the hub base. Although previous work by Kiner et al. formulated the dynamics to simulate N prescribed motion bodies that are not required to be directly connected to the rigid spacecraft hub,[3,29] the developed dynamics formulation requires the hub-relative states of all components to be known. Thus, even these prescribed motion effectors are not providing branching but hub-relative descriptions.

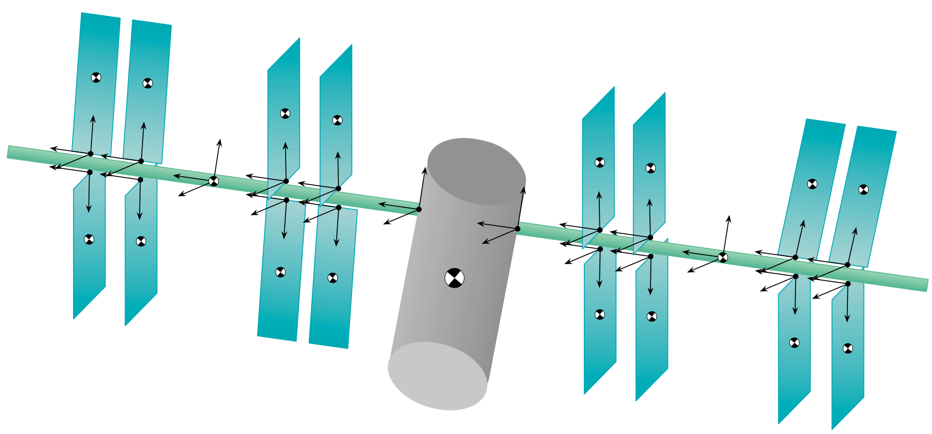

Figure 1. Prescribed solar array boom (green) with connected dynamic sub-components (blue).

This paper builds upon prior work in multi-body prescribed motion dynamics and expands the existing simulation space to capture hub-relative prescribed spacecraft components with branching behavior relative to the prescribed motion component. This will allow attached sub-components to dynamically move subject to the fully coupled system equations of motion. The Backsubstitution formulation enables a general, extendable implementation of this branching without having to re-derive the full equations of motion for each spacecraft configuration. Such a formulation will be useful to simulate actuated spacecraft components with flexible components such as the large deploying solar arrays of the ISS, Lucy, or DART. An example of this type of multi-body actuator is illustrated in Fig. (1), where individual flexing solar panels are attached to an articulated truss modeled as a rigid hub-relative prescribed motion component.

The organization of this paper is as follows. The following section provides an overview of the general spacecraft design of interest and presents the required frame definitions and parameters used for the dynamics derivation. Next, the spacecraft system translational and rotational dynamics are derived in the third section, followed by derivation of the sub-component equations of motion in the fourth section. The sub-component equations are then decoupled so that they may be readily substituted back into the system equations of motion. The fifth section organizes the full system equations of motion into the Backsubstitution Method[18–20] to facilitate a modular software implementation. The concluding remarks are offered in the final section of this paper.

PROBLEM STATEMENT

This work develops the equations of motion for a spacecraft system consisting of a rigid hub (gray) with an attached multi-body actuator component suitable for implementation with the Backsubstitution Method. The actuator component consists of a single hub-connected prescribed motion platform (green) that contains an attached dynamic sub-component (blue). The equations are derived with complete generality so that a wide variety of spacecraft concepts can be modeled and simulated using an identical derivation. The prescribed motion body actuates relative to the spacecraft hub, while the second-order states of the dynamic sub-component can be derived relative to the prescribed motion component. The modularity of the derived formulation ultimately enables N sub-components to be connected to the prescribed motion component and M of the described multi-body actuator components to be attached to the central spacecraft hub. The spacecraft geometry of interest for this derivation is illustrated in Fig. (2). Although only one dynamic sub-component is shown, the formulation can be similarly derived assuming N sub-components are connected generally to the prescribed motion platform.

There are five coordinate frames required for the system dynamics derivation. The dynamics are developed with respect to an inertial reference frame indicated by $\mathcal{N} : \{N, \hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$. The hub body frame $\mathcal{B} : \{B, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ describes the motion of the rigid spacecraft hub. The origin of this frame is located at a hub-fixed point B. The point B_c is defined as the center of mass of the hub, which is also body-fixed as a result of the rigid body assumption. Note that although points B and B_c are often assumed to coincide for a simpler dynamics formulation, they are kept as distinct locations in order to improve the ease of technical exchanges between different spacecraft mission teams. For example, a structures frame is often defined by the structural engineering team that is used to define the location of all the spacecraft components relative to a single fixed location on the spacecraft hub.

Next, the prescribed motion component body frame is designated by $\mathcal{P} : \{P, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{p}}_3\}$. This frame describes the motion of the prescribed body attached to the hub through the hub-fixed mount interface indicated by the frame $\mathcal{M} : \{M, \hat{\boldsymbol{m}}_1, \hat{\boldsymbol{m}}_2, \hat{\boldsymbol{m}}_3\}$. The origin of the prescribed motion body is located at the point P that is fixed to the prescribed body. The point P_c denotes the center of mass of the prescribed body. The mount frame is fixed with respect to the hub at the point M and is introduced as a matter of kinematic convenience. The prescribed body motion is profiled relative to this body-fixed frame to simplify the associated kinematic description.

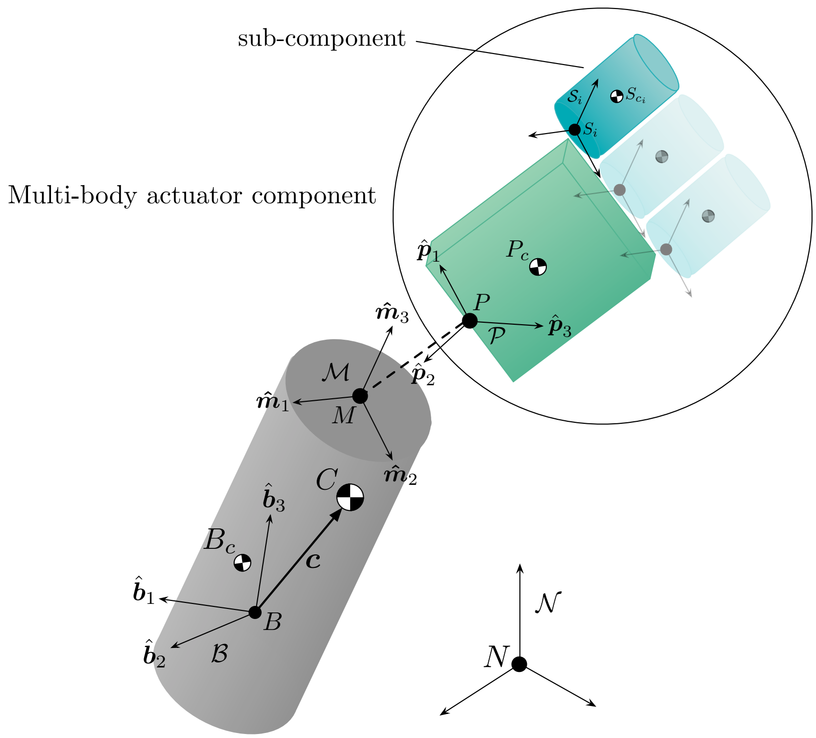

Figure 2. Spacecraft geometry, variables, and coordinate frames of interest.

Next, the dynamic sub-component frame is denoted $\mathcal{S} : \{S, \hat{\boldsymbol{s}}_1, \hat{\boldsymbol{s}}_2, \hat{\boldsymbol{s}}_3\}$, where the point S indicates the origin of the sub-component body frame and the point S_c is the center of mass point of the sub-component. Further, the vector $\boldsymbol{c}$ designates the center of mass location C of the entire spacecraft system relative to the hub frame origin point B. Note that for a completely general equation of motion formulation, the points $B, B_c, M, P, P_c, S, S_c$ and C are assumed to be not necessarily coincident.

The translational and rotational states of the prescribed motion body with respect to the spacecraft hub are assumed to be known and therefore prescribed in this derivation. These prescribed states are provided in Table (1). The left super-script indicates the frame of reference the parameters are expressed in, while the right $v^{'}$ superscript indicates a hub $\mathcal{B}$ frame relative time derivative. The attitude coordinates chosen to express the relative orientations between reference frames are Modified Rodriguez Parameters.[30] Because all bodies in this derivation are rigid, the center of mass locations of each body relative to the origin of each body's respective frame are constant. As a result of the hub-fixed orientation of the mount frame, the mount frame location and attitude relative to the hub frame are constant and its angular rates are zero.

EQUATIONS OF MOTION DERIVATION

This section uses Newtonian and Eulerian mechanics to derive the spacecraft hub translational and rotational equations of motion. The hub dynamics are derived rather than the system center of mass dynamics in order to leverage the Backsubstitution Method[18–20] for future software implementation and verification of the derived dynamics.

Table 1. **Prescribed motion component profiled states and fixed derivation parameters.**

Prescribed Motion Component States	Fixed Parameters
$\mathbf{r}_{P/M}$	${}^{\mathcal{B}}\mathbf{r}_{B_c/B}$
$\mathbf{r}'_{P/M}$	${}^{\mathcal{P}}\mathbf{r}_{P_c/P}$
$\mathbf{r}''_{P/M}$	${}^{\mathcal{S}}\mathbf{r}_{S_c/S}$
$\boldsymbol{\sigma}_{\mathcal{P}/\mathcal{M}}$	${}^{\mathcal{B}}\mathbf{r}_{M/B}$
$\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}$	$\boldsymbol{\sigma}_{\mathcal{M}/\mathcal{B}}$
$\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}}$	${}^{\mathcal{M}}\boldsymbol{\omega}_{\mathcal{M}/\mathcal{B}}$
	${}^{\mathcal{M}}\boldsymbol{\omega}'_{\mathcal{M}/\mathcal{B}}$

The Backsubstitution Method relies on a hub-centric spacecraft design configuration, where all dynamical components are attached to a central rigid body hub. This method has been shown to solve critical issues of software maintainability, scalability, and testability, while also achieving computational efficiency compared to other software tools. Further, this method eliminates the need to invert an entire system mass matrix that scales with the cube of the number of system states. Instead, the component-hub coupling is back-solved using only 3×3 matrix inversions. This drastically reduces the computational overhead to simulate complex spacecraft systems.

The dynamics of complex dynamical systems are often derived in the form $[\boldsymbol{M}]\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}, t)$, where $[\boldsymbol{M}]$ is the system mass matrix, $\boldsymbol{X}$ is the system state vector, $\dot{\boldsymbol{X}}$ is the time derivative of the state vector, and $\boldsymbol{f}(\boldsymbol{X}, t)$ is a function of the state vector and time. Instead, the Backsubstitution Method uses the following hub-centric form to modularize the equation of motion development:

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}} \end{bmatrix} \tag{1}$$

The hub second-order inertial translational acceleration $\ddot{\mathbf{r}}_{B/N}$ and inertial rotational angular velocity $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ states are separated from the other terms and are brought to the left-hand side of the expression. The translational and rotational equation terms coupled with the hub inertial translational acceleration are contained in the 3×3 $[A]$ and $[C]$ elements, respectively; while the translational and rotational terms coupled with the hub inertial angular acceleration are incorporated in the 3×3 $[B]$ and $[D]$ elements, respectively.

Translational Equations of Motion

Derivation of the spacecraft hub translational equations of motion begins by applying Newton's second law[30] to the spacecraft system center of mass point C:

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{C/N} = \boldsymbol{F}_{\text{ext}} \tag{2}$$

where m_{sc} is the total mass of the spacecraft system, $\boldsymbol{r}_{C/N}$ is the position vector of the system center of mass point relative to the inertial frame origin point N, and $\boldsymbol{F}_{\text{ext}}$ is the sum of all external forces acting on the system. The notation used for an inertial time derivative of a vector $\boldsymbol{v}$ is denoted by $\dot{\boldsymbol{v}}$ in this derivation.

Note that the acceleration of the hub frame origin point B must be defined in order to derive the hub translational equations of motion. Accordingly, the inertial acceleration of the spacecraft hub point B can be expressed as:

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \ddot{\boldsymbol{c}} \tag{3}$$

where $\boldsymbol{c}$ is the vector $\boldsymbol{r}_{C/B}$ that describes the location of the system center of mass point relative to the hub frame origin point. In order to find the inertial acceleration of the center of mass vector $\ddot{\boldsymbol{c}}$, first the transport theorem[30] is used to define the center of mass inertial velocity:

$$\dot{\boldsymbol{c}} = \boldsymbol{c}^{'} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} \tag{4}$$

The notation used for a hub (body) frame time derivative of a vector $\boldsymbol{v}$ is designated by $\boldsymbol{v}^{'}$ in this derivation. Next, taking the inertial time derivative of Eq. (4) gives the inertial acceleration of the center of mass vector:

$$\ddot{\boldsymbol{c}} = \dot{\boldsymbol{c}}^{'} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\boldsymbol{c}} \tag{5}$$

The hub-relative time derivative of Eq. (4) must be evaluated in order to define the $\dot{\boldsymbol{c}}^{'}$ term seen in Eq. (5):

$$\dot{\boldsymbol{c}}^{'} = \boldsymbol{c}^{''} + \left(\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}\right) + \left(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}^{'}\right) \tag{6}$$

Incorporating Eqs. (4) and (6) into Eq. (5) yields the expanded form:

$$\ddot{\boldsymbol{c}} = \boldsymbol{c}^{''} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}^{'} + [\dot{\tilde{\boldsymbol{\omega}}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} \tag{7}$$

Note that this derivation uses two notations for the cross-product operator: $\boldsymbol{v} \times \boldsymbol{w} = [\tilde{\boldsymbol{v}}]\boldsymbol{w}$ where $\boldsymbol{v} \times \boldsymbol{w}$ is the vector cross product and $[\tilde{\boldsymbol{v}}]\boldsymbol{w}$ is the vector cross product expressed in matrix form where $[\tilde{\boldsymbol{v}}]$ is a 3×3 skew-symmetric matrix and the components of $\boldsymbol{v}$ and $\boldsymbol{w}$ are expressed in the same frame.[30] Viewing Eq. (7), it is evident that both the hub-relative velocity $\boldsymbol{c}^{'}$ and acceleration $\boldsymbol{c}^{''}$ of the center of mass vector must be found in order to determine the inertial acceleration.

First, the center of mass vector is expressed using the mass contributions from the hub, prescribed motion body, and the sub-component:

$$\boldsymbol{c} = \frac{m_{\text{hub}}\boldsymbol{r}_{B_c/B} + m_{\text{P}}\boldsymbol{r}_{P_c/B} + m_{\text{S}}\boldsymbol{r}_{S_c/B}}{m_{\text{sc}}} \tag{8}$$

where

$$m_{\text{sc}} = m_{\text{hub}} + m_{\text{P}} + m_{\text{S}} \tag{9}$$

Next, the hub-relative velocity of the center of mass vector can be found:

$$\boldsymbol{c}^{'} = \frac{m_{\text{hub}}\boldsymbol{r}^{'}_{B_c/B} + m_{\text{P}}\boldsymbol{r}^{'}_{P_c/B} + m_{\text{S}}\boldsymbol{r}^{'}_{S_c/B}}{m_{\text{sc}}} \tag{10}$$

Using the rigid body assumption for the hub, Eq. (10) simplifies to

$$\boldsymbol{c}^{'} = \frac{m_{\text{P}}\boldsymbol{r}^{'}_{P_c/B} + m_{\text{S}}\boldsymbol{r}^{'}_{S_c/B}}{m_{\text{sc}}} \tag{11}$$

where

$$\boldsymbol{r}^{'}_{P_c/B} = \boldsymbol{r}^{'}_{P_c/P} + \boldsymbol{r}^{'}_{P/M} + \boldsymbol{r}^{'}_{M/B} \tag{12}$$

and

$$\boldsymbol{r}'_{S_c/B} = \boldsymbol{r}'_{S_c/P} + \boldsymbol{r}'_{P/M} + \boldsymbol{r}'_{M/B} \tag{13}$$

Recall that point M is considered hub-fixed in this work; therefore $\boldsymbol{r}'_{M/B} = \mathbf{0}$. Using the rigid body assumption and the transport theorem to simplify Eqs. (12) and (13) yields:

$$\boldsymbol{r}'_{P_c/B} = \boldsymbol{r}'_{P_c/M} = [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{r}_{P_c/P} + \boldsymbol{r}'_{P/M} \tag{14}$$

and

$$\boldsymbol{r}'_{S_c/B} = \boldsymbol{r}'_{S_c/M} = \frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{r}_{S_c/P} + \boldsymbol{r}'_{P/M} \tag{15}$$

Next, the hub-relative acceleration of the center of mass vector is found by taking the $\mathcal{B}$ frame time derivative of Eq. (11):

$$\boldsymbol{c}'' = \frac{m_{\text{P}}\boldsymbol{r}''_{P_c/M} + m_{\text{S}}\boldsymbol{r}''_{S_c/M}}{m_{\text{sc}}} \tag{16}$$

The product rule is applied to Eq. (14) to solve for $\boldsymbol{r}''_{P_c/M}$:

$$\boldsymbol{r}''_{P_c/M} = \left([\tilde{\boldsymbol{\omega}}'_{\mathcal{P}/\mathcal{M}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\right)\boldsymbol{r}_{P_c/P} + \boldsymbol{r}''_{P/M} \tag{17}$$

Similarly, solving for $\boldsymbol{r}''_{S_c/M}$:

$$\boldsymbol{r}''_{S_c/M} = \frac{^{\mathcal{P}}\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_{S_c/P} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} + \left([\tilde{\boldsymbol{\omega}}'_{\mathcal{P}/\mathcal{M}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\right)\boldsymbol{r}_{S_c/P} + \boldsymbol{r}''_{P/M} \tag{18}$$

Now that the inertial acceleration of the center of mass vector is known, Eq. (3) can now be expressed using Eq. (7):

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \boldsymbol{c}'' - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\dot{\tilde{\boldsymbol{\omega}}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} \tag{19}$$

The hub translational equations of motion are obtained by substituting Eq. (19) into Eq. (2):

$$m_{\text{sc}}\left(\ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{c}'' + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' + [\dot{\tilde{\boldsymbol{\omega}}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c}\right) = \boldsymbol{F}_{\text{ext}} \tag{20}$$

Incorporating Eq. (16) into Eq. (20) and bringing the second-order hub state variables to the left-hand side results in a system mass matrix-type form ideal for software implementation:

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} + m_{\text{sc}}[\dot{\tilde{\boldsymbol{\omega}}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} - m_{\text{P}}\boldsymbol{r}''_{P_c/M} - m_{\text{S}}\boldsymbol{r}''_{S_c/M} \tag{21}$$

Equation (21) gives the translational equations of motion for the spacecraft hub point B with respect to the inertial frame. This general, frame-independent vector equation enables flexible analysis of a broad range of spacecraft mission design configurations.

Rotational Equations of Motion

Next, the hub rotational equations of motion are derived. Separating the kinematic and kinetic differential equations enables convenient use of the angular velocity vector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ in the kinetic rotational equations of motion while not limiting the choice of attitude coordinates used to describe the kinematic orientation of the hub. The hub kinetic rotational equations of motion are derived starting from Euler's equation applied to the case where the spacecraft angular momentum is expressed about the hub-fixed point B not coincident with the system center of mass:[30]

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{L}_B + m_{\text{sc}}(\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c}) \tag{22}$$

where $\boldsymbol{H}_{\text{sc},B}$ is the inertial angular momentum of the spacecraft system about point B and $\boldsymbol{L}_B$ is the total external torque acting on the system about point B.

First, the system inertial angular momentum about point B must be defined:

$$\boldsymbol{H}_{\text{sc},B} = \boldsymbol{H}_{\text{hub},B} + \boldsymbol{H}_{\text{P},B} + \boldsymbol{H}_{\text{S},B} \tag{23}$$

The general definition of the hub angular momentum about point B is:

$$\boldsymbol{H}_{\text{hub},B} = \boldsymbol{H}_{\text{hub},B_c} + m_{\text{hub}}(\boldsymbol{r}_{B_c/B} \times \dot{\boldsymbol{r}}_{B_c/B}) \tag{24}$$

The hub inertial angular momentum about its center of mass point B_c is:

$$\boldsymbol{H}_{\text{hub},B_c} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{25}$$

where $[I_{\text{hub},B_c}]$ is the moment of inertia of the hub about its center of mass point. Because the hub is considered rigid in this derivation, Eq. (24) simplifies to the compact form:

$$\boldsymbol{H}_{\text{hub},B} = [I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{26}$$

where

$$[I_{\text{hub},B}] = [I_{\text{hub},B_c}] - m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}]^2 \tag{27}$$

Next, the angular momentum of the prescribed motion body about point B is:

$$\boldsymbol{H}_{\text{P},B} = \boldsymbol{H}_{\text{P},P_c} + m_{\text{P}}(\boldsymbol{r}_{P_c/B} \times \dot{\boldsymbol{r}}_{P_c/B}) \tag{28}$$

where

$$\dot{\boldsymbol{r}}_{P_c/B} = \boldsymbol{r}'_{P_c/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{P_c/B} \tag{29}$$

The inertial angular momentum of the prescribed body about its center of mass point P_c is:

$$\boldsymbol{H}_{\text{P},P_c} = [I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{N}} \tag{30}$$

Rewriting Eq. (28) using Eqs. (29) and (30) yields:

$$\boldsymbol{H}_{\text{P},B} = [I_{\text{P},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M} \tag{31}$$

Next, the angular momentum of the dynamic sub-component about point B is found:

$$\boldsymbol{H}_{\text{S},B} = \boldsymbol{H}_{\text{S},S_c} + m_{\text{S}}(\boldsymbol{r}_{S_c/B} \times \dot{\boldsymbol{r}}_{S_c/B}) \tag{32}$$

where

$$\dot{\boldsymbol{r}}_{S_c/B} = \boldsymbol{r}'_{S_c/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_c/B} \tag{33}$$

The inertial angular momentum of the sub-component about its center of mass point S_c is:

$$\boldsymbol{H}_{\text{S},S_c} = [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{34}$$

Rewriting Eq. (32) using Eqs. (33) and (34) yields:

$$\boldsymbol{H}_{\text{S},B} = [I_{\text{S},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M} \tag{35}$$

The total spacecraft inertial angular momentum expressed in Eq. (23) can now be rewritten by combining Eqs. (26), (31), and (35):

$$\begin{aligned}\boldsymbol{H}_{\text{sc},B} =& \left([I_{\text{hub},B}] + [I_{\text{P},B}] + [I_{\text{S},B}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ [I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M} \\ &+ [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\end{aligned} \tag{36}$$

To simplify Eq. (36), first the inertias of the rigid bodies about point B can be combined to yield the total spacecraft inertia about point B:

$$[I_{\text{sc},B}] = [I_{\text{hub},B}] + [I_{\text{P},B}] + [I_{\text{S},B}] \tag{37}$$

Equation (36) becomes:

$$\begin{aligned}\boldsymbol{H}_{\text{sc},B} = [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M} \\ +[I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\end{aligned} \tag{38}$$

The final task to develop the hub rotational equations of motion is to take the inertial time derivative of the system angular momentum about point B using the transport theorem:

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{H}'_{\text{sc},B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{H}_{\text{sc},B} \tag{39}$$

First, the hub $\mathcal{B}$ frame time derivative of Eq. (38) is given by:

$$\begin{aligned}\boldsymbol{H}'_{\text{sc},B} = [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \Big([I'_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + [I_{\text{P},P_c}]\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}''_{P_c/M}\Big) \\ + \Big([I'_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + [I_{\text{S},S_c}]\boldsymbol{\omega}'_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}''_{S_c/M}\Big)\end{aligned} \tag{40}$$

Using the inertia tensor transport theorem[31] to solve for the prescribed body and sub-component inertia derivatives using the rigid body assumption gives:

$$[I'_{\text{P},P_c}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}][I_{\text{P},P_c}] - [I_{\text{P},P_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}] \tag{41}$$

$$[I'_{\text{S},S_c}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{M}}][I_{\text{S},S_c}] - [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{M}}] \tag{42}$$

Solving for $[I'_{\text{sc},B}]$ using the rigid body assumption for the hub yields:

$$[I'_{\text{sc},B}] = [I'_{\text{P},B}] + [I'_{\text{S},B}] \tag{43}$$

where using the parallel axis theorem gives the inertia of the prescribed motion body and the sub-component about point B:

$$[I_{\text{P},B}] = [I_{\text{P},P_c}] + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}_{P_c/B}]^T \tag{44}$$

$$[I_{\text{S},B}] = [I_{\text{S},S_c}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/B}]^T \tag{45}$$

The $\mathcal{B}$ frame time derivatives of Eqs. (44) and (45) using the chain rule are given by:

$$[I'_{\text{P},B}] = [I'_{\text{P},P_c}] + m_{\text{P}}\left([\tilde{\boldsymbol{r}}'_{P_c/M}][\tilde{\boldsymbol{r}}_{P_c/B}]^T + [\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}'_{P_c/M}]^T\right) \tag{46}$$

$$[I'_{\text{S},B}] = [I'_{\text{S},S_c}] + m_{\text{S}}\left([\tilde{\boldsymbol{r}}'_{S_c/M}][\tilde{\boldsymbol{r}}_{S_c/B}]^T + [\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}'_{S_c/M}]^T\right) \tag{47}$$

Incorporating the above results yields the full expression for $[I'_{\text{sc},B}]$

$$\begin{aligned}[I'_{\text{sc},B}] = [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}][I_{\text{P},P_c}] - [I_{\text{P},P_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}] + m_{\text{P}}\left([\tilde{\boldsymbol{r}}'_{P_c/M}][\tilde{\boldsymbol{r}}_{P_c/B}]^T + [\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}'_{P_c/M}]^T\right)\\ [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{M}}][I_{\text{S},S_c}] - [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{M}}] + m_{\text{S}}\left([\tilde{\boldsymbol{r}}'_{S_c/M}][\tilde{\boldsymbol{r}}_{S_c/B}]^T + [\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}'_{S_c/M}]^T\right)\end{aligned} \tag{48}$$

Next, evaluating the transport term in Eq. (39) gives the result:

$$\begin{aligned}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{H}_{\text{sc},B} = {} & [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M}\\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\end{aligned} \tag{49}$$

Substituting Eqs. (40) and (49) into Eq. (39) and equating the result with Eq. (22) gives the following intermediate result for the hub rotational equations of motion:

$$\begin{aligned}[I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} & + [I'_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + [I_{\text{P},P_c}]\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}''_{P_c/M}\\ & + [I'_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + [I_{\text{S},S_c}]\boldsymbol{\omega}'_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}''_{S_c/M}\\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{P},P_c}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + m_{\text{P}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M}\\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}} + m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\\ & = \boldsymbol{L}_B - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N}\end{aligned} \tag{50}$$

Bringing the hub second-order states to the left-hand side and grouping similar terms yields the final form of the hub rotational equations of motion:

$$\begin{aligned}m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_B & - \left([I'_{\text{sc},B}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\ & - \left([I'_{\text{P},P_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{P},P_c}]\right)\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}\\ & - \left([I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{M}}\\ & - m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}''_{P_c/M} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}''_{S_c/M}\\ & - [I_{\text{P},P_c}]\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} - [I_{\text{S},S_c}]\boldsymbol{\omega}'_{\mathcal{S}/\mathcal{M}}\\ & - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\right)\end{aligned} \tag{51}$$

Indeed, the rotational equations of motion confirm the aforementioned dynamic coupling between the spacecraft system components. The rotational dynamics of the hub are clearly influenced by both the translational and rotational states of the sub-components, as seen on the right-hand side of the expression. Note that because the prescribed motion body's hub-relative states are completely prescribed, no differential equations need to be integrated for these components. The hub-relative prescribed states $\boldsymbol{r}_{P/M}, \boldsymbol{r}'_{P/M}, \boldsymbol{r}''_{P/M}, \boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}, \boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}}$ and $\boldsymbol{\sigma}_{\mathcal{P}/\mathcal{M}}$ are assumed to be known and accordingly kinematically profiled at each instant in time during all phases of the simulated spacecraft motion. Therefore, there are 12 coupled kinetic differential equations needed to fully define the hub motion.

SUB-COMPONENT EQUATIONS OF MOTION

Sub-Component Translational Equations of Motion

The sub-component translational equations of motion are derived starting with Newton's second law applied to the sub-component's center of mass point S_c:

$$m_{\text{S}}\ddot{\boldsymbol{r}}_{S_c/N} = \boldsymbol{F}_{\text{ext}} \tag{52}$$

where the sub-component inertial translational acceleration can be written as:

$$\ddot{\boldsymbol{r}}_{S_c/N} = \ddot{\boldsymbol{r}}_{S_c/P} + \ddot{\boldsymbol{r}}_{P/N} \tag{53}$$

The first inertial time derivative of $\boldsymbol{r}_{S_c/P}$ can be written using the transport theorem with the prescribed motion body frame $\mathcal{P}$:

$$\dot{\boldsymbol{r}}_{S_c/P} = \frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{r}_{S_c/P} \tag{54}$$

Similarly, the second inertial time derivative is:

$$\ddot{\boldsymbol{r}}_{S_c/P} = \frac{^{\mathcal{P}}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{r}}_{S_c/P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} \tag{55}$$

Substituting Eq. (55) into Eq. (53) followed by substitution of the result into Eq. (52) gives the translational equation of motion for the sub-component:

$$m_{\text{S}}\left(\frac{^{\mathcal{P}}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{r}}_{S_c/P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} + \ddot{\boldsymbol{r}}_{P/N}\right) = \boldsymbol{F}_{\text{ext}} \tag{56}$$

Re-arranging Eq. (56) by bringing the sub-component's second-order states to the left-hand side gives the final form of the sub-component translational equations of motion:

$$\begin{aligned} m_{\text{S}}\frac{^{\mathcal{P}}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} = &- m_{\text{S}}\ddot{\boldsymbol{r}}_{P/N} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} \\ &+ \boldsymbol{F}_{\text{ext}} - 2m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} \end{aligned} \tag{57}$$

Sub-Component Rotational Equations of Motion

Next, the sub-component rotational equations of motion are developed. Because the sub-component is a rigid body, the angular momentum of the sub-component about its body-frame origin point S is given by:

$$\boldsymbol{H}_{\text{S},S} = [I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{58}$$

Equation (58) can be written by introducing the prescribed motion body frame $\mathcal{P}$ as:

$$\boldsymbol{H}_{\text{S},S} = [I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{N}} \tag{59}$$

Next, taking the inertial time derivative of Eq. (59) using the transport theorem with the prescribed motion body frame yields:

$$\dot{\boldsymbol{H}}_{\text{S},S} = \frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{H}_{\text{S},S} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{H}_{\text{S},S} \tag{60}$$

The $\mathcal{P}$ frame time derivative of Eq. (59) is:

$$\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{H}_{\text{S},S} = \frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}[I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [I_{\text{S},S}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [I_{\text{S},S}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} \tag{61}$$

Equation (60) becomes:

$$\dot{\boldsymbol{H}}_{\text{S},S} = \frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}[I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [I_{\text{S},S}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [I_{\text{S},S}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}][I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{62}$$

To further develop the sub-component's rotational equations of motion, Euler's equation is next applied about point S:

$$\dot{\boldsymbol{H}}_{\text{S},S} = \boldsymbol{L}_S - m_{\text{S}}(\boldsymbol{r}_{S_c/S} \times \ddot{\boldsymbol{r}}_{S/N}) \tag{63}$$

Equation (63) can be expanded as:

$$\dot{\boldsymbol{H}}_{\text{S},S} = \boldsymbol{L}_S - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\left(\ddot{\boldsymbol{r}}_{S/S_c} + \ddot{\boldsymbol{r}}_{S_c/P} + \ddot{\boldsymbol{r}}_{P/N}\right) \tag{64}$$

where

$$\dot{\boldsymbol{r}}_{S/S_c} = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]\boldsymbol{r}_{S/S_c} \tag{65}$$

and similarly

$$\ddot{\boldsymbol{r}}_{S/S_c} = [\dot{\tilde{\boldsymbol{\omega}}}_{\mathcal{S}/\mathcal{N}}]\boldsymbol{r}_{S/S_c} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S/S_c} \tag{66}$$

The inertial time derivative of $\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}}$ is given by:

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{P}} + \dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} = \frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} \tag{67}$$

Rewriting Eq. (66) using Eq. (67) yields the result:

$$\ddot{\boldsymbol{r}}_{S/S_c} = \left(\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}\right) \times \boldsymbol{r}_{S/S_c} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S/S_c} \tag{68}$$

Equation (64) can now be re-written using the results from Eq. (55) and (68):

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{S},S} = \boldsymbol{L}_S - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\Bigg\{&\left(\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}\right) \times \boldsymbol{r}_{S/S_c} + [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S/S_c}\\ &+ \frac{{}^{\mathcal{P}}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{r}}_{S_c/P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} + \ddot{\boldsymbol{r}}_{P/N}\Bigg\}\end{aligned} \tag{69}$$

Finally, substitution of Eq. (62) into the left-hand side of Eq. (69) with some re-arranging of terms gives the rotational equations of motion for the sub-component:

$$\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}[I_{\mathrm{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [I_{\mathrm{S},S}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [I_{\mathrm{S},S}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}][I_{\mathrm{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} = \boldsymbol{L}_S \\ - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\Bigg\{[\tilde{\boldsymbol{r}}_{S_c/S}]\left(\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}\right) - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/S} \\ + \frac{^{\mathcal{P}}\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_{S_c/P} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{r}}_{S_c/P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} + \ddot{\boldsymbol{r}}_{P/N}\Bigg\} \tag{70}$$

Bringing the sub-component's second-order states to the left-hand side of Eq. (70) results in the system mass-matrix form of the sub-component rotational equations of motion:

$$m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\frac{^{\mathcal{P}}\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_{S_c/P} + [I_{\mathrm{S},S_c}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} = - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\ddot{\boldsymbol{r}}_{P/N} \\ - \left([I_{\mathrm{S},S_c}] - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}][\tilde{\boldsymbol{r}}_{S_c/P}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} \\ + \boldsymbol{L}_S - 2m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} \\ - \left(\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}[I_{\mathrm{S},S}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}][I_{\mathrm{S},S}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \\ - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]^2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\ + m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/S} - [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P}\right) \tag{71}$$

Decoupling The Sub-Component Equations of Motion

Next, the sub-component equations of motion must be decoupled in order to incorporate the sub-component's second-order states into the system equations of motion. This approach eliminates the explicit dependency on the sub-component equations of motion from the system equations of motion.

First, the sub-component equations of motion presented in Eqs. (57) and (71) are written in the coupled system-mass matrix form given by:

$$[\boldsymbol{M}_{6\times6}]\begin{bmatrix}\frac{^{\mathcal{P}}\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_{S_c/P} \\ \frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}}\end{bmatrix} = [\boldsymbol{N}^*_{6\times3}]\ddot{\boldsymbol{r}}_{P/N} + [\boldsymbol{P}^*_{6\times3}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\boldsymbol{Q}^*_{6\times1}] \tag{72}$$

where

$$[\boldsymbol{M}_{6\times6}] = \begin{bmatrix} m_{\mathrm{S}}[I_{3\times3}] & [0_{3\times3}] \\ m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}] & [I_{\mathrm{S},S_c}]\end{bmatrix} \tag{73}$$

$$[\boldsymbol{N}^*_{6\times3}] = \begin{bmatrix} -m_{\mathrm{S}}[I_{3\times3}] \\ -m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\end{bmatrix} \tag{74}$$

$$[\boldsymbol{P}^*_{6\times3}] = \begin{bmatrix} m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/P}] \\ -\left([I_{\mathrm{S},S_c}] - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}][\tilde{\boldsymbol{r}}_{S_c/P}]\right)\end{bmatrix} \tag{75}$$

$$[\boldsymbol{Q}^*_{6\times1}] = \begin{bmatrix} \boldsymbol{F}_{\text{ext}} - 2m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P} \\ \\ \boldsymbol{L}_S - 2m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/S}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} \\ -\left(\frac{^\mathcal{P}\text{d}}{\text{d}t}[I_{\text{S},S}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}][I_{\text{S},S}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \\ -m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]^2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\ +m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\left([\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/S} - [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}}]^2\boldsymbol{r}_{S_c/P}\right) \end{bmatrix} \tag{76}$$

Inverting the mass matrix to solve for the sub-component second-order states yields the result:

$$\begin{bmatrix} \frac{^\mathcal{P}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} \\ \frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \end{bmatrix} = [\boldsymbol{N}]\ddot{\boldsymbol{r}}_{P/N} + [\boldsymbol{P}]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\boldsymbol{Q}] \tag{77}$$

where

$$[\boldsymbol{N}] = [\boldsymbol{M}]^{-1}[\boldsymbol{N}^*] \tag{78}$$

$$[\boldsymbol{P}] = [\boldsymbol{M}]^{-1}[\boldsymbol{P}^*] \tag{79}$$

$$[\boldsymbol{Q}] = [\boldsymbol{M}]^{-1}[\boldsymbol{Q}^*] \tag{80}$$

The sub-component second-order states are therefore given by:

$$\frac{^\mathcal{P}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} = [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{P/N} + [\boldsymbol{P}_1]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\boldsymbol{Q}_1] \tag{81}$$

$$\frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} = [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{P/N} + [\boldsymbol{P}_2]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{N}} + [\boldsymbol{Q}_2] \tag{82}$$

Next, Eqs. (81) and (82) must be written in terms of the hub second-order states:

$$\frac{^\mathcal{P}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} = [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{B/N} + [\boldsymbol{P}_1]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{P/B} + [\boldsymbol{P}_1]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}} + [\boldsymbol{Q}_1] \tag{83}$$

$$\frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} = [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{B/N} + [\boldsymbol{P}_2]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{P/B} + [\boldsymbol{P}_2]\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}} + [\boldsymbol{Q}_2] \tag{84}$$

where

$$\ddot{\boldsymbol{r}}_{P/B} = \boldsymbol{r}''_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{r}}_{P/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B} \tag{85}$$

$$\dot{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}} = \boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \tag{86}$$

Finally, substituting Eqs. (85) and (86) into Eqs. (83) and (84) gives the desired form of the sub-component second-order states to be substituted into the system equations of motion:

$$\begin{aligned} \frac{^\mathcal{P}\text{d}^2}{\text{d}t^2}\boldsymbol{r}_{S_c/P} = [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{B/N} &+ \left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ &+ [\boldsymbol{N}_1]\left(\boldsymbol{r}''_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ &+ [\boldsymbol{P}_1]\left(\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}\right) + [\boldsymbol{Q}_1] \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{^\mathcal{P}\text{d}}{\text{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} = [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{B/N} &+ \left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ &+ [\boldsymbol{N}_2]\left(\boldsymbol{r}''_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ &+ [\boldsymbol{P}_2]\left(\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}\right) + [\boldsymbol{Q}_2] \end{aligned} \tag{88}$$

BACKSUBSTITUTION METHOD IMPLEMENTATION

Finally, the system equations of motion can be written in the Backsubstitution formulation form given by:

$$\begin{bmatrix}[\boldsymbol{A}] & [\boldsymbol{B}] \\ [\boldsymbol{C}] & [\boldsymbol{D}]\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}}\end{bmatrix} \tag{89}$$

First, however, note that the $\boldsymbol{r}^{''}_{S_c/M}$ and $\boldsymbol{\omega}^{'}_{\mathcal{S}/\mathcal{M}}$ terms seen in Eqs. (21) and (51) must be expanded in order to incorporate Eqs. (87) and (88) into the system equations of motion. Using the transport theorem, $\boldsymbol{r}^{''}_{S_c/M}$ is determined in Eq. (18) and $\boldsymbol{\omega}^{'}_{\mathcal{S}/\mathcal{M}}$ is given by:

$$\boldsymbol{\omega}^{'}_{\mathcal{S}/\mathcal{M}} = \frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}} \tag{90}$$

Rewriting Eqs. (18) and (90) using Eqs. (87) and (88) yields:

$$\begin{aligned}\boldsymbol{r}^{''}_{S_c/M} = {} & [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{B/N} + \left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ & + [\boldsymbol{N}_1]\left(\boldsymbol{r}^{''}_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^{'}_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ & + [\boldsymbol{P}_1]\left(\boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}\right) + [\boldsymbol{Q}_1] \\ & + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} + \left([\tilde{\boldsymbol{\omega}}^{'}_{\mathcal{P}/\mathcal{M}}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\right)\boldsymbol{r}_{S_c/P} + \boldsymbol{r}^{''}_{P/M}\end{aligned} \tag{91}$$

$$\begin{aligned}\boldsymbol{\omega}^{'}_{\mathcal{S}/\mathcal{M}} = {} & [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{B/N} + \left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ & + [\boldsymbol{N}_2]\left(\boldsymbol{r}^{''}_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^{'}_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ & + [\boldsymbol{P}_2]\left(\boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}}\right) + [\boldsymbol{Q}_2] \\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + \boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}}\end{aligned} \tag{92}$$

Combining similar terms in the above expressions gives the following results:

$$\begin{aligned}\boldsymbol{r}^{''}_{S_c/M} = {} & [\boldsymbol{N}_1]\ddot{\boldsymbol{r}}_{B/N} + \left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ & + \left([\boldsymbol{N}_1] + [I_{3\times3}]\right)\boldsymbol{r}^{''}_{P/M} \\ & + \left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right)\boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}} \\ & + [\boldsymbol{N}_1]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^{'}_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ & + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P} \\ & + [\boldsymbol{P}_1][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} + [\boldsymbol{Q}_1]\end{aligned} \tag{93}$$

$$\begin{aligned}\boldsymbol{\omega}^{'}_{\mathcal{S}/\mathcal{M}} = {} & [\boldsymbol{N}_2]\ddot{\boldsymbol{r}}_{B/N} + \left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ & + [\boldsymbol{N}_2]\left(\boldsymbol{r}^{''}_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^{'}_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\ & + \left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\boldsymbol{\omega}^{'}_{\mathcal{P}/\mathcal{M}} \\ & + [\boldsymbol{P}_2][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\ & + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} + [\boldsymbol{Q}_2]\end{aligned} \tag{94}$$

Note that only Eq. (93) must be substituted into the system translational equations of motion while both Eqs. (93) and (94) must be substituted into the system rotational equations of motion. Integration of Eqs. (93) and (94) into Eq. (51) can be improved by first combining the following relevant terms given in Eq. (51):

$$\begin{aligned}
-m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}''_{S_c/M} - [I_{\text{S},S_c}]\boldsymbol{\omega}'_{\mathcal{S}/\mathcal{M}} = &-\left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\ddot{\boldsymbol{r}}_{B/N} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{N}_1] + [I_{3\times3}]\right)\boldsymbol{r}''_{P/M} \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- 2m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P} \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{P}_1][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{Q}_1] \\
&- [I_{\text{S},S_c}][\boldsymbol{N}_2]\left(\boldsymbol{r}''_{P/M} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- [I_{\text{S},S_c}][\boldsymbol{P}_2][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\
&- [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} - [I_{\text{S},S_c}][\boldsymbol{Q}_2]
\end{aligned} \tag{95}$$

Simplifying Eq. (95) by combining similar terms yields:

$$\begin{aligned}
-m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}''_{S_c/M} - [I_{\text{S},S_c}]\boldsymbol{\omega}'_{\mathcal{S}/\mathcal{M}} = &-\left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\ddot{\boldsymbol{r}}_{B/N} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{N}_1] + [I_{3\times3}]\right) + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\boldsymbol{r}''_{P/M} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P}\right) \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{P}_1] + [I_{\text{S},S_c}][\boldsymbol{P}_2]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\
&- [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{Q}_1] - [I_{\text{S},S_c}][\boldsymbol{Q}_2]
\end{aligned} \tag{96}$$

Next, substituting Eq. (91) into the system translational equations of motion and substituting Eq.

(96) into the system rotational equations of motion gives the following expressions:

$$
\begin{aligned}
(m_{\text{sc}}[I_{3\times3}] + m_{\text{S}}[\boldsymbol{N}_1])\, \ddot{\boldsymbol{r}}_{B/N} + \left(m_{\text{S}}[\boldsymbol{P}_1] + m_{\text{S}}[\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}] - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\right) \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} \\
- 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} \\
- m_{\text{P}}\boldsymbol{r}''_{P_c/M} - m_{\text{S}}\left([\boldsymbol{N}_1] + [I_{3\times3}]\right)\boldsymbol{r}''_{P/M} \\
- m_{\text{S}}\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
- m_{\text{S}}[\boldsymbol{N}_1]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
- 2m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P} \\
- m_{\text{S}}[\boldsymbol{P}_1][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} - m_{\text{S}}[\boldsymbol{Q}_1]
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
&\left(m_{\text{sc}}[\tilde{\boldsymbol{c}}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\ddot{\boldsymbol{r}}_{B/N} \\
&+ \left([I_{\text{sc},B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&= \boldsymbol{L}_B - \left([I'_{\text{sc},B}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&- \left([I'_{\text{P},P_c}] + [I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{\text{P},P_c}] + [I_{\text{S},S_c}]\right) + \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{P}_1] + [I_{\text{S},S_c}][\boldsymbol{P}_2]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\right)\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\
&- \left([I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}] + [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\
&- m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}''_{P_c/M} \\
&- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}]\boldsymbol{r}'_{P_c/M} + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\boldsymbol{r}'_{S_c/M}\right) \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{N}_1] + [I_{3\times3}]\right) + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\boldsymbol{r}''_{P/M} \\
&- \left([I_{\text{P},P_c}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P}\right) \\
&- m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{Q}_1] - [I_{\text{S},S_c}][\boldsymbol{Q}_2]
\end{aligned} \tag{98}
$$

The final step in obtaining the system equations of motion requires expanding the terms $\boldsymbol{r}''_{P_c/M}$, $\boldsymbol{r}'_{P_c/M}$ and $\boldsymbol{r}'_{S_c/M}$ given by Eqs. (14), (15), and (17):

$$
\begin{aligned}
(m_{\text{sc}}[I_{3\times3}] + m_{\text{S}}[\boldsymbol{N}_1])\, \ddot{\boldsymbol{r}}_{B/N} + \left(m_{\text{S}}[\boldsymbol{P}_1] + m_{\text{S}}[\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}] - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\right) \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} \\
- 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} \\
- \left(m_{\text{P}}[I_{3\times3}] + m_{\text{S}}\left([\boldsymbol{N}_1] + [I_{3\times3}]\right)\right)\boldsymbol{r}''_{P/M} \\
- \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/P}] + m_{\text{S}}\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
- m_{\text{S}}[\boldsymbol{N}_1]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
- 2m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\left(m_{\text{S}}\boldsymbol{r}_{S_c/P} - m_{\text{P}}\boldsymbol{r}_{P_c/P}\right) \\
- m_{\text{S}}[\boldsymbol{P}_1][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} - m_{\text{S}}[\boldsymbol{Q}_1]
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
&\left(m_{\text{sc}}[\tilde{\boldsymbol{c}}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\ddot{\boldsymbol{r}}_{B/N} \\
&\quad + \left([I_{\text{sc},B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right)\right)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&= \boldsymbol{L}_B - \left([I'_{\text{sc},B}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&\quad - \Big\{[I'_{\text{P},P_c}] + [I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{\text{P},P_c}] + [I_{\text{S},S_c}] - m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}_{P_c/P}] - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/P}]\right) \\
&\quad + \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{P}_1] + [I_{\text{S},S_c}][\boldsymbol{P}_2]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big\}\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\
&\quad - \left([I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}] + [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\
&\quad - \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{N}_1] + [I_{3\times3}]\right) + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\boldsymbol{r}''_{P/M} \\
&\quad + \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}_{P_c/P}] - [I_{\text{P},P_c}] - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right) - [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&\quad - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\right)\boldsymbol{r}'_{P/M} \\
&\quad - \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&\quad - m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{P_c/P} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P}\right) \\
&\quad - m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_c/B}]\frac{{}^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{Q}_1] - [I_{\text{S},S_c}][\boldsymbol{Q}_2]
\end{aligned} \tag{100}
$$

Finally, expressing the equations of motion above in the Backsubstitution form given in Eq. (89) yields the final form of the system equations of motion for software implementation. The Backsubstitution contributions are given by:

$$[\boldsymbol{A}] = m_{\text{sc}}[I_{3\times3}] + m_{\text{S}}[\boldsymbol{N}_1] \tag{101}$$

$$[\boldsymbol{B}] = m_{\text{S}}[\boldsymbol{P}_1] + m_{\text{S}}[\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}] - m_{\text{sc}}[\tilde{\boldsymbol{c}}] \tag{102}$$

$$[\boldsymbol{C}] = m_{\text{sc}}[\tilde{\boldsymbol{c}}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2] \tag{103}$$

$$[\boldsymbol{D}] = [I_{\text{sc},B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] + [\boldsymbol{N}_1][\tilde{\boldsymbol{r}}_{P/B}]\right) + [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [\boldsymbol{N}_2][\tilde{\boldsymbol{r}}_{P/B}]\right) \tag{104}$$

$$
\begin{aligned}
\mathbf{v}_{\text{trans}} = \boldsymbol{F}_{\text{ext}} &- 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{c} \\
&- \left(m_{\text{P}}[I_{3\times3}] + m_{\text{S}}\left([\boldsymbol{N}_1] + [I_{3\times3}]\right)\right)\boldsymbol{r}''_{P/M} \\
&- \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/P}] + m_{\text{S}}\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- m_{\text{S}}[\boldsymbol{N}_1]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- 2m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{S_c/P} - [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\left(m_{\text{S}}\boldsymbol{r}_{S_c/P} - m_{\text{P}}\boldsymbol{r}_{P_c/P}\right) \\
&- m_{\text{S}}[\boldsymbol{P}_1][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} - m_{\text{S}}[\boldsymbol{Q}_1]
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
\mathbf{v}_{\text{rot}} = \boldsymbol{L}_B &- \left([I'_{\text{sc},B}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&- \Big\{[I'_{\text{P},P_c}] + [I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{\text{P},P_c}] + [I_{\text{S},S_c}] - m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}_{P_c/P}] - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/P}]\right) \\
&+ \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{P}_1] + [I_{\text{S},S_c}][\boldsymbol{P}_2]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big\}\boldsymbol{\omega}_{\mathcal{P}/\mathcal{M}} \\
&- \left([I'_{\text{S},S_c}] + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{S},S_c}] + [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\right)\boldsymbol{\omega}_{\mathcal{S}/\mathcal{P}} \\
&- \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{N}_1] + [I_{3\times3}]\right) + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\boldsymbol{r}''_{P/M} \\
&+ \left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{r}}_{P_c/P}] - [I_{\text{P},P_c}] - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left([\boldsymbol{P}_1] - [\tilde{\boldsymbol{r}}_{S_c/P}]\right) - [I_{\text{S},S_c}]\left([\boldsymbol{P}_2] + [I_{3\times3}]\right)\right)\boldsymbol{\omega}'_{\mathcal{P}/\mathcal{M}} \\
&- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\right)\boldsymbol{r}'_{P/M} \\
&- \left(m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{N}_1] + [I_{\text{S},S_c}][\boldsymbol{N}_2]\right)\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P/M} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^2\boldsymbol{r}_{P/B}\right) \\
&- m_{\text{P}}[\tilde{\boldsymbol{r}}_{P_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{P_c/P} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}]\left(2[\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} + [\tilde{\boldsymbol{\omega}}_{\mathcal{P}/\mathcal{M}}]^2\boldsymbol{r}_{S_c/P}\right) \\
&- m_{\text{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_c/B}]\frac{{}^{\mathcal{P}}\text{d}}{\text{d}t}\boldsymbol{r}_{S_c/P} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\boldsymbol{Q}_1] - [I_{\text{S},S_c}][\boldsymbol{Q}_2]
\end{aligned}
\tag{106}
$$

CONCLUSION

Dynamic modeling and simulation of complex space vehicles is essential to support the launch of any space mission. Especially as spacecraft concepts continue to grow in complexity with diverse types of appendages being attached to the central spacecraft hub, the ability to simulate these spacecraft systems becomes crucial for mission success.

Advancements in spacecraft appendages such as deployable solar array booms containing M connected solar panels or M multi-link robotic arms are nontrivial to model and simulate correctly. Inspired by these complex spacecraft structures, this work thoroughly derives the generalized equations of motion for a spacecraft consisting of a rigid hub with an attached two-body actuator component. The actuator component contains a hub-connected prescribed hub-relative motion platform and a rigid dynamic sub-component that is attached to the prescribed motion platform. The modularity of the derived formulation ultimately enables N dynamic sub-components to be attached to the prescribed motion body and further, M of the described multi-body actuator components to be attached to the central spacecraft hub. The derived equations are useful to describe large hub-relative deployable structures that contain other attached dynamic components.

Future work involves software verification of the derived equations of motion using angular momentum and energy conservation principles.

REFERENCES

[1] B. G. Özdemir, "Power stage design and implementation of a deploymentmechanism driver for space applications," Master's thesis, Middle East Technical University, 2012.

[2] J. Sun, "A New Design of the 3D Deployable Solar Array in the Aerospace Field and Kinematic Analysis during Deployment," *Journal of Physics: Conference Series*, Vol. 2364, nov 2022, p. 012039, 10.1088/1742-6596/2364/1/012039.

[3] L. Kiner, C. Allard, and H. Schaub, "Prescribed Motion Dynamics For Spacecraft Solar Array Deployment," *AAS Guidance and Control Conference*, Breckenridge, CO, Feb. 2–7 2024. Paper No. AAS 24-123.

[4] M. K. Chamberlain, S. H. Kiefer, M. LaPointe, and P. LaCorte, "On-orbit flight testing of the Roll-Out Solar Array," *Acta Astronautica*, Vol. 179, 2021, pp. 407–414.

[5] R. Calaon, L. Kiner, C. Allard, and H. Schaub, “Momentum Management Of A Spacecraft Equipped With A Dual-Gimballed Electric Thruster,” *AAS Guidance and Control Conference*, Breckenridge, CO, Feb. 2–8 2023. Paper No. AAS-23-178.

[6] J. R. Brophy, “NASA’s Deep Space 1 Ion Engine (Plenary),” *Review of Scientific Instruments*, Vol. 73, No. 2, 2002, pp. 1071–1078.

[7] B. N. M. M. H. J. Brophy, C. Garner and D. Noon, “General Hinged Rigid-Body Dynamics Approximating First-Order Spacecraft Solar Panel Flexing,” *The Ion Propulsion System for Dawn*, Vol. 55, 2003, p. 4542.

[8] T. D. W. H. T. I. e. a. D. Y. Oh, S. Collins, “Development of the Psyche Mission for NASA’s Discovery Program,” *36th International Electric Propulsion Conference*, University of Vienna, Vienna Austria, Sep. 15–20 2019. Paper No. IEPC-2019-192.

[9] J. S. Sovey, V. K. Rawlin, and M. J. Patterson, “Ion Propulsion Development Projects in U.S.: Space Electric Rocket Test I to Deep Space 1,” *Journal of Propulsion and Power*, Vol. 17, No. 3, 2001, pp. 517–526, 10.2514/2.5806.

[10] R. Calaon, C. Allard, and H. Schaub, “Continuous Center-Of-Mass Estimation For A Gimbaled Ion-Thruster Equipped Spacecraft,” *AAS/AIAA Astrodynamics Specialist Conference*, Big Sky, MO, Aug. 13–17 2023. Paper AAS 23–218.

[11] L. Kiner, C. Allard, and H. Schaub, “Multi-Body Prescribed Spacecraft Dynamics Subject To Actuator Inputs,” *AAS/AIAA Astrodynamics Specialist Conference*, Big Sky, MT, Aug. 13–17 2023. Paper AAS 23–230.

[12] R. G. D. B. A. Aikenhead and F. M. Davis, “Canadarm and The Space Shuttle,” *Journal of Vacuum Science and Technology A: Vacuum, Surfaces, and Films*, Vol. 1, No. 2, 1983, pp. 126–132.

[13] S. Sachdev, “Canadarm- a review of its flights,” *Journal of Vacuum Science and Technology A: Vacuum, Surfaces, and Films*, Vol. 4, No. 3, 1986, pp. 268–272.

[14] K. B. M. Hiltz, C. Rice and R. Allison, “Canadarm: 20 years of mission success through adaptation,” *International Symposium on Artificial Intelligence, Robotics and Automation*, University of Vienna, Vienna Austria, 2001. Paper No. JSC-CN-6877.

[15] “Multibody Dynamics Module (online database), COMSOL,” 2024. `https://www.comsol.com/multibody-dynamics-module` [retrieved Apr. 2024].

[16] “Adams: The Multibody Dynamics Simulation Solution (online database), MSC Software,” 2024. `http://www.mscsoftware.com/product/adams` [retrieved Apr. 2024].

[17] “SimscapeMultibody:Model and Simulate Multibody Mechanical Systems (online database), MathWorks,” 2024. `https://www.mathworks.com/products/simmechanics.html` [retrieved Apr. 2024].

[18] C. Allard, M. D. Ramos, H. Schaub, P. Kenneally, and S. Piggott, “Modular Software Architecture for Fully Coupled Spacecraft Simulations,” *Journal of Aerospace Information Systems*, Vol. 15, No. 12, 2018, pp. 670–683.

[19] P. W. Kenneally, S. Piggott, and H. Schaub, “Basilisk: A Flexible, Scalable and Modular Astrodynamics Simulation Framework,” *Journal of Aerospace Information Systems*, Vol. 17, Sept. 2020, pp. 496–507.

[20] C. Allard, M. D. Ramos, and H. Schaub, “Computational Performance of Complex Spacecraft Simulations Using Back-Substitution,” *Journal of Aerospace Information Systems*, Vol. 16, No. 10, 2019, pp. 427–436.

[21] C. Allard, H. Schaub, and S. Piggott, “General Hinged Rigid-Body Dynamics Approximating First-Order Spacecraft Solar Panel Flexing,” *Journal of Spacecraft and Rockets*, Vol. 55, No. 5, 2018, pp. 1290–1298.

[22] C. Allard, M. Diaz-Ramos, and H. Schaub, “Spacecraft Dynamics Integrating Hinged Solar Panels and Lumped-Mass Fuel Slosh Model,” *AIAA/AAS Astrodynamics Specialist Conference*, Long Beach, CA, Sept. 12–15 2016.

[23] J. Alcorn, C. Allard, and H. Schaub, “Fully Coupled Reaction Wheel Static and Dynamic Imbalance for Spacecraft Jitter Modeling,” *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 6, 2018, pp. 1380–1388, 10.2514/1.G003277.

[24] J. Alcorn, C. Allard, and H. Schaub, “Fully-Coupled Dynamical Jitter Modeling Of Variable-Speed Control Moment Gyroscopes,” *AAS/AIAA Astrodynamics Specialist Conference*, Stevenson, WA, Aug. 20–24 2017. Paper No. AAS-17-730.

[25] J. Vaz Carneiro, C. Allard, and H. Schaub, “Rotating Rigid Body Dynamics Architecture For Spacecraft Simulation Software Implementation,” *AAS Guidance and Control Conference*, Breckenridge, CO, Feb. 2–8 2023. Paper No. AAS-23-112.

[26] J. Vaz Carneiro, C. Allard, and H. Schaub, “Effector Dynamics For Sequentially Rotating Rigid Body Spacecraft Components,” *AAS/AIAA Astrodynamics Specialist Conference*, Big Sky, MO, Aug. 13–17 2023. Paper AAS 23–192.

[27] J. Vaz Carneiro, P. Johnson, and H. Schaub, “Backsubstitution Method For Spacecraft With Generally Translating Appendages,” *AAS Astrodynamics Specialist Conference*, Broomfield, CO, Aug. 11–15 2024. Paper No. AAS 24-248.

[28] G. Bascom, L. Kiner, and H. Schaub, “Spacecraft Dynamics Analysis Using Stochastic Point-Mass Model of Human Motion,” AAS/AIAA Space Flight Mechanics Meeting, Austin, TX, Jan. 15–19, 2023.

[29] L. Kiner, J. V. Carneiro, C. Allard, and H. Schaub, “Spacecraft Simulation Software Implementation of General Prescribed Motion Dynamics of Two Connected Rigid Bodies,” AAS Rocky Mountain GN&C Conference, Breckenridge, CO, Feb. 2–8, 2023.

[30] H. Schaub and J. L. Junkins, *Analytical Mechanics of Space Systems*. Reston, Virginia: American Institute of Aeronautics and Astronautics, Inc., 4 ed., 2018.

[31] C. Allard, J. Maxwell, and H. Schaub, “A Transport Theorem for the Inertia Tensor for Simplified Spacecraft Dynamics Development,” International Astronautical Conference, Paris, France, Sept. 18-22, 2022.