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A Transport Theorem for the Inertia Tensor for Simplified Spacecraft Dynamics Development

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Abstract

Dynamics development for free floating bodies in space involves describing how the body moves and can be complex for multi-body systems. If the body is assumed rigid, the change in mass properties is trivial because they do not change with respect to the body. However, if this is assumption cannot be made, describing the change in mass properties is critical. A key aspect of this variation is the change in the inertia tensor of the composite spacecraft. The transport theorem of vectors is widely utilized for dynamics development but is not well suited for simplification of the expression for the time rate of change of the inertia tensor. In this paper, the transport theorem of the inertia tensor is derived. A dynamics derivation is included and highlights the simplifications and time savings gained by the application of this tool.

1. Introduction

Spacecraft are becoming increasingly complex with potentially multiple moving components on board. Some examples of these articulating bodies on spacecraft are solar arrays, high gain antennas, robotic manipulators, and instrument payloads. Depending on the size of these components, the impact on the spacecraft from the relative motion of the articulating bodies can be substantial. This necessitates the ability to model the physics behind this phenomenon.

When deriving the equations of motion of motion for spacecraft dynamics, the use of the Transport Theorem[1] for vectors is common place. However, when deriving equations of motions for multi-body spacecraft systems, using the vector based Transport Theorem for the inertia tensor requires extensive algebra. A Transport Theorem for the inertia tensor is introduced in this paper and extends the work seen in References.[2,3]

The utility of the inertia Transport Theorem is highlighted by an example provided in this paper and is compared to using the vector based Transport Theorem. The application of this theorem is broad for multi-body dynamics.

2. Derivation of Inertia Transport Theorem

Figure 1 shows an arbitrary body moving with respect to an inertial coordinate system $\mathcal{N}$. For generality, assume the body is non-rigid and has variable density across its volume. A non-inertial coordinate system $\mathcal{B}$ is coincident with the time-varying body Center-Of-Mass (COM) located at position $\boldsymbol{R}_{\text{c}}$. The body rotates with angular velocity $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ relative to the inertial frame. All possible deformations of the body are contained within the volumetric boundary V shown.

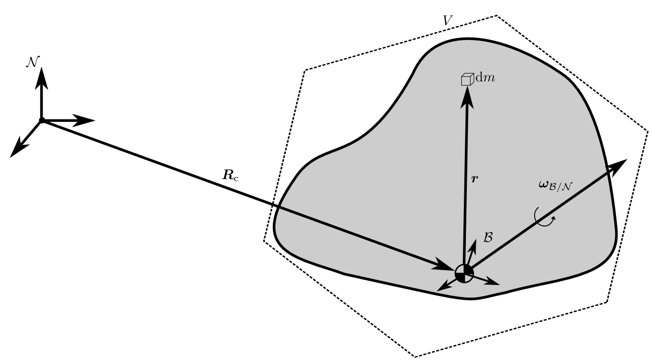

Fig. 1: Arbitrary rotating body

A differential mass element $\mathrm{d}m$ is located at position $\boldsymbol{r}$ relative to the body COM. The angular velocity $\boldsymbol{H}_{\text{c}}$ of the body about its COM is calculated according to Eq. 1. The method presented below works as well for an arbitrary, non-COM reference point, but the resulting terms and complications are immaterial to the purpose of this paper.

$$\boldsymbol{H}_{\mathrm{c}} = \int_V \boldsymbol{r} \times \dot{\boldsymbol{r}} \mathrm{d}m \tag{1}$$

Substituting the well-known Transport Theorem,[1] the inertial derivative ${}^{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \dot{\boldsymbol{r}}$ is calculated in terms of the $\mathcal{B}$-frame derivative ${}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \boldsymbol{r}'$.

$$\boldsymbol{H}_{\mathrm{c}} = \int_V \boldsymbol{r} \times (\boldsymbol{r}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}) \mathrm{d}m \tag{2}$$

Distributing the integral over the summation yields a two-term expression for the body angular momentum.

$$\begin{aligned} \boldsymbol{H}_{\mathrm{c}} &= \int_V \boldsymbol{r} \times \boldsymbol{r}' \mathrm{d}m + \int_V \boldsymbol{r} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}) \mathrm{d}m \\ &= \boldsymbol{H}_{\mathrm{c}_{\mathrm{def}}} + \boldsymbol{H}_{\mathrm{c}_{\mathrm{rot}}} \end{aligned} \tag{3}$$

The first term in Eq. 3 accounts for angular momentum from deformation of the body and is calculable as-is given a model for how the body deforms. The second term is the angular momentum resulting from the body undergoing rotational motion. This is the term of focus.

$$\boldsymbol{H}_{\mathrm{c}_{\mathrm{rot}}} = \int_V \boldsymbol{r} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}) \mathrm{d}m \tag{4}$$

The cross product is a linear operator and can therefore be represented as a matrix. In this case, this matrix is skew-symmetric, giving rise the the cross product's anti-commutative nature.

$$\begin{aligned} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r} &= [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r} = -[\tilde{\boldsymbol{r}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ [\tilde{\boldsymbol{r}}] &= \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \end{aligned} \tag{5}$$

Substituting Eq. 5 into 4 and taking the mass-independent angular velocity outside the integral yields the well-known expression for the rotating-body angular momentum.

$$\boldsymbol{H}_{\mathrm{c}_{\mathrm{rot}}} = -\int_V [\tilde{\boldsymbol{r}}]^2 \mathrm{d}m \, \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = [I_{\mathrm{c}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{6}$$

The inertia tensor equation is determined by inspection of Eq. 6.

$$[I_{\mathrm{c}}] = -\int_V [\tilde{\boldsymbol{r}}]^2 \mathrm{d}m \tag{7}$$

The dynamics of the arbitrary body shown in Figure 1 is calculated through the well known equation

$$\dot{\boldsymbol{H}}_{\mathrm{c}} = \dot{\boldsymbol{H}}_{\mathrm{c}_{\mathrm{def}}} + \dot{\boldsymbol{H}}_{\mathrm{c}_{\mathrm{rot}}} = \boldsymbol{L}_{\mathrm{c}} \tag{8}$$

where $\boldsymbol{L}_{\mathrm{c}}$ is the torque on the body about its COM. Assuming models for deformations of and torques on the body, only the rotating angular momentum component's derivative remains unknown.

$$\dot{\boldsymbol{H}}_{\mathrm{c}_{\mathrm{rot}}} = [\dot{I}_{\mathrm{c}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{9}$$

The latter term in Eq. 9 is recognized as the rigid-body angular momentum component and can be evaluated via integration of the equations of motion in Eq. 8. The first term, however, presents a significant challenge. In most cases, the modeled body is assumed to consist of multiple rigid bodies undergoing relative motion. This enables calculation of $[\dot{I}_{\mathrm{c}}]$ as a sum of contributions from each rigid body translated to the body COM via parallel axis theorem. For an arbitrary shape like that in Figure 1, this simplification cannot be applied. Instead, an expression for $[\dot{I}_{\mathrm{c}}]$ must be determined.

The inertia expression shown in Eq. 7 does not immediately offer any promise. The only obviously practical solution is to evaluate term-by-term once the inertia is expressed in a frame. However, this numerical approach precludes the use of convenient linear algebra tools that massively simplify dynamics derivations and analyses. Instead, an identity is derived that enables derivation of a simple expression for $[\dot{I}_{\mathrm{c}}]$.

The cross product operator shown in Eq. 5 is squared to yield an element-by-element expression for the integrand in Eq. 7.

$$[\tilde{\boldsymbol{r}}]^2 = \begin{bmatrix} -r_2^2 - r_3^2 & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & -r_1^2 - r_3^2 & r_2 r_3 \\ r_1 r_3 & r_2 r_3 & -r_1^2 - r_2^2 \end{bmatrix} \tag{10}$$

Important to note in this equation is that no frame has been specified, meaning that the elements r_i are indeterminate. Applying the magnitude identity $-r_k^2 - r_j^2 = r_i^2 - r^2$ yields a vector equation for the inertia integrand.

$$\begin{aligned} [\tilde{\boldsymbol{r}}]^2 &= \begin{bmatrix} r_1^2 - r^2 & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & r_2^2 - r^2 & r_2 r_3 \\ r_1 r_3 & r_2 r_3 & r_3^2 - r^2 \end{bmatrix} \\ &= \boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{r}^T\boldsymbol{r}[I_{3\times 3}] \end{aligned} \tag{11}$$

Here, $[I_{3\times 3}]$ indicates the three-dimensional identity matrix. Substituting this vector equation into Eq. 7 and differentiating yields

$$\begin{aligned} [\dot{I}_{\mathrm{c}}] &= -\,{}^{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t} \int_V [\tilde{\boldsymbol{r}}]^2 \mathrm{d}m \\ &= -\,{}^{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t} \int_V \left(\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{r}^T\boldsymbol{r}[I_{3\times 3}]\right) \mathrm{d}m \end{aligned} \tag{12}$$

The three operators in Eq. 12 (differentiation, integration, and summation) are all linear, which allows for commutation of the first two operators and distribution of the differentiation over the sum. This enables focus on the integrand rather than the full expression.

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 = \left(\dot{\boldsymbol{r}}\boldsymbol{r}^T + \boldsymbol{r}\dot{\boldsymbol{r}}^T - 2\boldsymbol{r}^T\dot{\boldsymbol{r}}[I_{3\times3}]\right) \tag{13}$$

Another substitution of transport theorem provides an expression for the integrand's inertial-frame derivative expressed in terms of $\mathcal{B}$-frame derivatives.

$$\begin{aligned}\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 &= (\boldsymbol{r}' + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r})\boldsymbol{r}^T \\ &+ \boldsymbol{r}(\boldsymbol{r}' + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r})^T - 2\boldsymbol{r}^T(\boldsymbol{r}' + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r})[I_{3\times3}])\end{aligned} \tag{14}$$

Eq. 14 is further simplified by distributing the outer and inner products over the sums and collecting terms containing $\boldsymbol{r}'$.

$$\begin{aligned}\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 &= \boldsymbol{r}'\boldsymbol{r}^T + \boldsymbol{r}\boldsymbol{r}'^T - 2\boldsymbol{r}^T\boldsymbol{r}'[I_{3\times3}]) \\ &+ \left([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}\boldsymbol{r}^T + \boldsymbol{r}\boldsymbol{r}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T - 2\boldsymbol{r}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}\right)\end{aligned} \tag{15}$$

The first bracketed term is recognized as the $\mathcal{B}$-frame derivative of the integrand. The last term in the second bracket is zero because the cross product is orthogonal to both its arguments. At this stage, it is necessary add the zero matrix to the equation in the form $[0_{3\times3}] = -\boldsymbol{r}^T\boldsymbol{r}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] + \boldsymbol{r}^T\boldsymbol{r}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^T\boldsymbol{r} - \boldsymbol{r}^T\boldsymbol{r}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T$. In this equality, the commutivity of scalars and skew-symmetric nature of the cross product are leveraged for simplicity.

$$\begin{aligned}\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 &= \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 + \left([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}\boldsymbol{r}^T - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}^T\boldsymbol{r}\right) \\ &\quad + \left(\boldsymbol{r}\boldsymbol{r}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T - \boldsymbol{r}^T\boldsymbol{r}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T\right) \\ &= \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left(\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{r}^T\boldsymbol{r}[I_{3\times3}]\right) \\ &\quad + \left(\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{r}^T\boldsymbol{r}[I_{3\times3}]\right)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T \\ &= \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}[\tilde{\boldsymbol{r}}]^2 + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}]^2 + [\tilde{\boldsymbol{r}}]^2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T\end{aligned} \tag{16}$$

Inserting this equation into Eq. 12 and performing minor operations yields

$$\begin{aligned}[\dot{I}_{\mathrm{c}}] &= -\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\int_V [\tilde{\boldsymbol{r}}]^2\mathrm{d}m \\ &- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\int_V [\tilde{\boldsymbol{r}}]^2\mathrm{d}m - \int_V [\tilde{\boldsymbol{r}}]^2\mathrm{d}m\,[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T\end{aligned} \tag{17}$$

Finally, the transport theorem for the inertia tensor is presented.[2,3] Note, the choice of frames is arbitrary.

$$[\dot{I}_{\mathrm{c}}] = [I_{\mathrm{c}}]' + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{c}}] + [I_{\mathrm{c}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T \tag{18}$$

Plugging this solution into Eq. 9 and applying a rigid body assumption (i.e. $[I_{\mathrm{c}}]' = [0]$) yields the well-known angular momentum derivative.

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{c_{rot}}} &= [I_{\mathrm{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \left([\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{c}}] + [I_{\mathrm{c}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &= [I_{\mathrm{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{c}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\end{aligned} \tag{19}$$

3. Spacecraft Dynamics Derivation Example

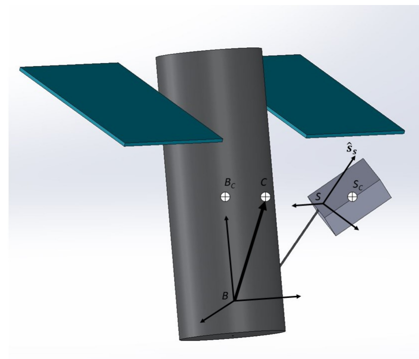

Fig. 2: Diagram of spacecraft with a single spinning body

A spacecraft dynamics example of a rigid hub connected to a single rigid spinning body is included to show the utility of the inertia Transport Theorem. The development considers the body frame and the spinning body frame. The body frame is denoted $\mathcal{B}$. The basis vectors of the body frame are

$$\mathcal{B} : \{B, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\} \tag{20}$$

The spinning body frame is denoted $\mathcal{S}$. Parameters relating to the spacecraft hub are denoted with a subscript text B. The hub and spinning body are allowed center of mass offsets from their respective coordinate frame origins. The hub's center of mass location is labeled as B_c. This location is described with respect to the body frame origin as $\boldsymbol{r}_{Bc/B}$. The spinning body is also allowed a general center of mass offset from the spinning body frame origin. This location is labeled as S_c and is located with respect to the spinning body frame origin as $\boldsymbol{r}_{S_c/S}$. The time-varying center of mass of the entire system is denoted C, and the vector from the body frame origin to point C is labeled $\boldsymbol{c}$. The spinning body rotates with respect to the hub about a body frame fixed axis $\hat{\boldsymbol{s}}_s$ with an angular velocity of Ω.

For this paper, the spacecraft rotational EOM and spinning body EOM is derived, but for completeness, the spacecraft translational EOM is included:

$$\begin{aligned}m_{\mathrm{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\mathrm{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - m_{\mathrm{S}}[\tilde{\boldsymbol{r}}_{S_c/S}]\hat{\boldsymbol{s}}\dot{\Omega}_{\mathrm{S}} \\ = \boldsymbol{F} - 2m_{\mathrm{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\mathrm{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \\ - m_{\mathrm{S}}[\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\boldsymbol{r}'_{S_c/S}\end{aligned} \tag{21}$$

The derivation of rotational EOM starts with the angular momentum of the spacecraft about point B.

$$\boldsymbol{H}_{\text{sc},B} = \boldsymbol{H}_{\text{hub},B} + \boldsymbol{H}_{\text{S},B} \tag{22}$$

The inertial time derivative of angular momentum when the body fixed coordinate frame origin is not coincident with the center of mass of the body is

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{L}_B + m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} \tag{23}$$

where $\boldsymbol{L}_B$ is the vector sum of external torques acting on the spacecraft. Differentiating Eq. (22), and using the Transport Theorem,[1] the inertial derivative of the spacecraft angular momentum is expressed as

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{H}'_{\text{sc},B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H}_{\text{sc},B} \tag{24}$$

Starting with the definition of the angular momentum of the hub about point B

$$\boldsymbol{H}_{\text{hub},B} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times \dot{\boldsymbol{r}}_{B_c/B} \tag{25}$$

Applying the Transport Theorem for $\dot{\boldsymbol{r}}_{B_c/B}$ yields

$$\begin{aligned}\boldsymbol{H}_{\text{hub},B} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B})\end{aligned} \tag{26}$$

$\boldsymbol{H}_{\text{hub},B}$ can be further simplified to the following expression

$$\boldsymbol{H}_{\text{hub},B} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{27}$$

The parallel axis theorem for the inertia of the hub about point B is

$$[I_{\text{hub},B}] = [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T \tag{28}$$

Using the expression for $[I_{\text{hub},B}]$, $\boldsymbol{H}_{\text{hub},B}$ can be further simplified to

$$\boldsymbol{H}_{\text{hub},B} = [I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{29}$$

Following a similar pattern, the following derivation is completed for the spinning body:

$$\begin{aligned}\boldsymbol{H}_{\text{S},B} = [I_{\text{S},S_c}](\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}}) \\ + m_{\text{S}}\boldsymbol{r}_{S_c/B} \times (\boldsymbol{r}'_{S_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_c/B})\end{aligned} \tag{30}$$

which simplifies to

$$\boldsymbol{H}_{\text{S},B} = [I_{\text{S},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + m_{\text{S}}\boldsymbol{r}_{S_c/B} \times \boldsymbol{r}'_{S_c/B} \tag{31}$$

Combining the two expression for angular momentum in Eq. (22) yields

$$\boldsymbol{H}_{\text{sc},B} = [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} + m_{\text{S}}\boldsymbol{r}_{S_c/B} \times \boldsymbol{r}'_{S_c/B} \tag{32}$$

Taking the body frame relative time derivative of $\boldsymbol{H}_{\text{sc},B}$ yields

$$\begin{aligned}\boldsymbol{H}'_{\text{sc},B} = [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \\ + [I_{\text{S},S_c}]\dot{\Omega}\hat{\boldsymbol{s}}_s + m_{\text{S}}\boldsymbol{r}_{S_c/B} \times \boldsymbol{r}''_{S_c/B}\end{aligned} \tag{33}$$

The inertia of the spinning body about point B is defined as

$$[I_{\text{S},B}] = [I_{\text{S},S_c}] + m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/B}]^T \tag{34}$$

With those descriptions, the body frame relative time derivative of the inertia of $[I_{\text{S},B}]$ can be defined as

$$[I_{\text{S},B}]' = [I_{\text{S},S_c}]' + m_{\text{S}}\left([\tilde{\boldsymbol{r}}'_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/B}]^T + [\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}'_{S_c/B}]^T\right) \tag{35}$$

Using the transport theorem of the inertia tensor, the computation of $[I_{\text{S},S_c}]'$ simply yields

$$[I_{\text{S},S_c}]' = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}][I_{\text{S},S_c}] + [I_{\text{S},S_c}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]^T \tag{36}$$

Combining the individual expressions into the single angular momentum expression and combining like terms:

$$\begin{aligned}m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I_{\text{S},S_c}]\hat{\boldsymbol{s}}_s\dot{\Omega} \\ + m_{\text{S}}\boldsymbol{r}_{S_c/B} \times \boldsymbol{r}''_{S_c/B} = \boldsymbol{L}_B - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{S},S_c}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} \\ - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H}_{\text{sc},B}\end{aligned} \tag{37}$$

Pull out second order derivatives for spinning body and rearranging yields

$$\begin{aligned}m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \Big([I_{\text{S},S_c}] \\ - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/S}]\Big)\hat{\boldsymbol{s}}_s\dot{\Omega} = \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{H}_{\text{sc},B} \\ - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{S},S_c}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\boldsymbol{r}'_{S_c/S}\end{aligned} \tag{38}$$

$[I_{\text{S},S_c}]'$ can be replaced to further simplify and arrive at the final form of the rotation EOM.

$$\begin{aligned}m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \Big([I_{\text{S},S_c}] \\ - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{r}}_{S_c/S}]\Big)\hat{\boldsymbol{s}}_s\dot{\Omega} = \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{H}_{\text{sc},B} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}][I_{\text{S},S_c}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{B}} - m_{\text{S}}[\tilde{\boldsymbol{r}}_{S_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]\boldsymbol{r}'_{S_c/S}\end{aligned} \tag{39}$$

Finally, the spinning body EOM starts with the modified Euler's equation:

$$\dot{\boldsymbol{H}}_{\text{S},S} = \boldsymbol{L}_S + m_{\text{S}}\ddot{\boldsymbol{r}}_{S/N} \times \boldsymbol{r}_{S_c/S} \tag{40}$$

The angular momentum of the spinner about point S is:

$$\boldsymbol{H}_{\text{S},S} = [I_{\text{S},S}]\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \tag{41}$$

Finding the inertial derivative of the previous expression yields

$$\dot{\boldsymbol{H}}_{\mathrm{S},S} = [I_{\mathrm{S},S}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [I_{\mathrm{S},S}]\big(\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_{\mathrm{S}}\hat{\boldsymbol{s}}\big) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H}_{\mathrm{S},S} \tag{42}$$

Using the inertia transport theorem, $[I_{\mathrm{S},S}]'$ is simply

$$[I_{\mathrm{S},S}]' = [\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}][I_{\mathrm{S},S}] + [I_{\mathrm{S},S}][\tilde{\boldsymbol{\omega}}_{\mathcal{S}/\mathcal{B}}]^T \tag{43}$$

Which completes the expression for $\dot{\boldsymbol{H}}_{\mathrm{S},S}$

$$\dot{\boldsymbol{H}}_{\mathrm{S},S} = [I_{\mathrm{S},S}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{S},S}]\hat{\boldsymbol{s}}\dot{\Omega}_{\mathrm{S}} + [I_{\mathrm{S},S}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{H}_{\mathrm{S},S} \tag{44}$$

Finally, the dynamics of the spinning body can be simplified to:

$$\begin{aligned}[I_{\mathrm{S},S}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [I_{\mathrm{S},S}]\hat{\boldsymbol{s}}\dot{\Omega}_{\mathrm{S}} + [I_{\mathrm{S},S}]'\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} \\ + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] = \boldsymbol{L}_S + m_{\mathrm{S}}\ddot{\boldsymbol{r}}_{S/N} \times \boldsymbol{r}_{S_c/S}\end{aligned} \tag{45}$$

The complexity of deriving equations of motion for multi-body spacecraft dynamics using only the vector Transport Theorem can be seen in Reference.[4]

4. Alternate Derivation of Inertia Transport Theorem

This derivation uses the basis vector definition of the inertia tensor to derive the transport theorem.

$$\begin{aligned}[I] = I_{11}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1^T + I_{12}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_2^T + I_{13}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_3^T + I_{21}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_1^T \\ + I_{22}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2^T + I_{23}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_3^T + I_{31}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_1^T + I_{32}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_2^T + I_{33}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3^T\end{aligned} \tag{46}$$

Taking the inertial derivative of $[I]$ yields

$$\begin{aligned}[\dot{I}] = \dot{I}_{11}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1^T + \dot{I}_{12}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_2^T + \dot{I}_{13}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_3^T + \dot{I}_{21}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_1^T \\ + \dot{I}_{22}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2^T + \dot{I}_{23}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_3^T + \dot{I}_{31}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_1^T + \dot{I}_{32}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_2^T + \dot{I}_{33}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3^T \\ + I_{11}\dot{\hat{\boldsymbol{b}}}_1\hat{\boldsymbol{b}}_1^T + I_{12}\dot{\hat{\boldsymbol{b}}}_1\hat{\boldsymbol{b}}_2^T + I_{13}\dot{\hat{\boldsymbol{b}}}_1\hat{\boldsymbol{b}}_3^T + I_{21}\dot{\hat{\boldsymbol{b}}}_2\hat{\boldsymbol{b}}_1^T \\ + I_{22}\dot{\hat{\boldsymbol{b}}}_2\hat{\boldsymbol{b}}_2^T + I_{23}\dot{\hat{\boldsymbol{b}}}_2\hat{\boldsymbol{b}}_3^T + I_{31}\dot{\hat{\boldsymbol{b}}}_3\hat{\boldsymbol{b}}_1^T + I_{32}\dot{\hat{\boldsymbol{b}}}_3\hat{\boldsymbol{b}}_2^T + I_{33}\dot{\hat{\boldsymbol{b}}}_3\hat{\boldsymbol{b}}_3^T \\ + I_{11}\hat{\boldsymbol{b}}_1\dot{\hat{\boldsymbol{b}}}_1^T + I_{12}\hat{\boldsymbol{b}}_1\dot{\hat{\boldsymbol{b}}}_2^T + I_{13}\hat{\boldsymbol{b}}_1\dot{\hat{\boldsymbol{b}}}_3^T + I_{21}\hat{\boldsymbol{b}}_2\dot{\hat{\boldsymbol{b}}}_1^T \\ + I_{22}\hat{\boldsymbol{b}}_2\dot{\hat{\boldsymbol{b}}}_2^T + I_{23}\hat{\boldsymbol{b}}_2\dot{\hat{\boldsymbol{b}}}_3^T + I_{31}\hat{\boldsymbol{b}}_3\dot{\hat{\boldsymbol{b}}}_1^T + I_{32}\hat{\boldsymbol{b}}_3\dot{\hat{\boldsymbol{b}}}_2^T + I_{33}\hat{\boldsymbol{b}}_3\dot{\hat{\boldsymbol{b}}}_3^T\end{aligned} \tag{47}$$

Combining body frame relative time derivatives simplifies the expression to

$$\begin{aligned}[\dot{I}] = [I]' + I_{11}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)\hat{\boldsymbol{b}}_1^T + I_{12}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)\hat{\boldsymbol{b}}_2^T + \\ I_{13}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)\hat{\boldsymbol{b}}_3^T + I_{21}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)\hat{\boldsymbol{b}}_1^T \\ + I_{22}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)\hat{\boldsymbol{b}}_2^T + I_{23}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)\hat{\boldsymbol{b}}_3^T \\ + I_{31}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)\hat{\boldsymbol{b}}_1^T + I_{32}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)\hat{\boldsymbol{b}}_2^T + I_{33}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)\hat{\boldsymbol{b}}_3^T \\ + I_{11}\hat{\boldsymbol{b}}_1(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)^T + I_{12}\hat{\boldsymbol{b}}_1(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)^T + \\ I_{13}\hat{\boldsymbol{b}}_1(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)^T + I_{21}\hat{\boldsymbol{b}}_2(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)^T \\ + I_{22}\hat{\boldsymbol{b}}_2(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)^T + I_{23}\hat{\boldsymbol{b}}_2(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)^T \\ + I_{31}\hat{\boldsymbol{b}}_3(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)^T + I_{32}\hat{\boldsymbol{b}}_3(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_2)^T + I_{33}\hat{\boldsymbol{b}}_3(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_3)^T\end{aligned} \tag{48}$$

The following expressions are useful for simplification

$$\begin{aligned}I_{11}(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)\hat{\boldsymbol{b}}_1^T + I_{11}\hat{\boldsymbol{b}}_1(\boldsymbol{\omega} \times \hat{\boldsymbol{b}}_1)^T = \\ I_{11}\,{}^{\mathcal{B}}\!\begin{bmatrix} 0 & 0 & 0 \\ \omega_3 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{bmatrix} + I_{11}\,{}^{\mathcal{B}}\!\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{49}$$

$$\begin{aligned}[\tilde{\boldsymbol{\omega}}]\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1^T &= {}^{\mathcal{B}}\!\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} {}^{\mathcal{B}}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= {}^{\mathcal{B}}\!\begin{bmatrix} 0 & 0 & 0 \\ \omega_3 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{bmatrix}\end{aligned} \tag{50}$$

Using these operations and combining like terms yield

$$\begin{aligned}[\dot{I}] = [I]' + [\tilde{\boldsymbol{\omega}}]\Big(I_{11}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1^T + I_{12}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_2^T + I_{13}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_3^T + I_{21}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_1^T \\ + I_{22}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2^T + I_{23}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_3^T \\ + I_{31}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_1^T + I_{32}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_2^T + I_{33}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3^T\Big) \\ + \Big(I_{11}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1^T + I_{12}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_2^T + I_{13}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_3^T + I_{21}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_1^T \\ + I_{22}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2^T + I_{23}\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_3^T \\ + I_{31}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_1^T + I_{32}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_2^T + I_{33}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3^T\Big)[\tilde{\boldsymbol{\omega}}]^T\end{aligned} \tag{51}$$

which can be simplified to the inertia tensor transport theorem

$$[\dot{I}_{\mathrm{c}}] = [I_{\mathrm{c}}]' + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathrm{c}}] + [I_{\mathrm{c}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]^T \tag{52}$$

5. Conclusion

The Transport Theorem for the inertia tensor is derived and the utility is highlighted by showing the derivation for a spacecraft dynamics example. Using this theorem, the required algebra is much less and the resulting formulation is simpler. This inertia Transport Theorem can be used broadly for spacecraft dynamics development and is general, allowing it be applied to more complex spacecraft dynamics problems.

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