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FULLY-COUPLED DYNAMICAL JITTER MODELING OF VARIABLE-SPEED CONTROL MOMENT GYROSCOPES

John Alcorn,* Cody Allard,† and Hanspeter Schaub‡

Control moment gyroscopes (CMGs) and variable-speed control moment gyroscopes (VSCMGs) are a popular method for spacecraft attitude control and fine pointing. However, since these devices typically operate at high wheel speeds, mass imbalances within the wheels act as a primary source of angular jitter. Although these effects are often characterized through experimentation in order to validate pointing stability requirements, it is of interest to include jitter in a computer simulation of the spacecraft in the early stages of spacecraft development. An estimate of jitter amplitude may be found by modeling imbalance torques as external disturbance forces and torques on the spacecraft. In this case, mass imbalances are lumped into static and dynamic imbalance parameters, allowing jitter force and torque to be simply proportional to wheel speed squared. A physically realistic dynamic model may be obtained by defining mass imbalances in terms of a wheel center of mass location and inertia tensor. The fully-coupled dynamic model allows for momentum and energy validation of the system. This is often critical when modeling additional complex dynamical behavior such as flexible dynamics and fuel slosh. This paper presents a generalized approach to VSCMG imbalance modeling of a rigid spacecraft hub with N VSCMGs. Implementations of the fully-coupled VSCMG model derived within this paper are released open-source as part of the Basilisk astrodynamics software.

INTRODUCTION

Control moment gyroscopes (CMGs) and variable-speed control moment gyroscopes (VSCMGs) are a popular method to control larger spacecraft. By nature, a CMG and VSCMG are fundamentally the same device – both trade angular momentum with the spacecraft hub by gimbaling a spinning flywheel. Unlike CMGs, VSCMGs are able to leverage the additional degree of freedom in the rate of rotation of the flywheel whereas a CMG keeps wheel speed near constant. CMGs and VSCMGs have multiple benefits over reaction wheels (RWs). These devices are typically more power efficient because they only require a torque on the gimbal axis to actuate. The wheel motor must simply maintain a constant wheel speed after the initial spin up of the wheel. Thus, CMGs/VSCMGs are particularly good at delivering large torques very quickly. However, a minimum of three devices are still required for a full 3D control solution. A cluster of CMGs can encounter singularities which prevent torque about certain axes and can lead to loss of control. VSCMGs can use cleverly devised control strategies to allow singularity avoidance by combining wheel speed changes and gimbal

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rates. "Null-motion" reconfiguration allows a cluster of VSCMGs to reconfigure without applying a net torque to the spacecraft hub.[1]

A problem with using any type of momentum exchange device (MED) for attitude control is that they cause vibration or "jitter" due to mass imbalances in the flywheels.[2–4] This is of greater concern when using CMGs or VSCMGs since they typically operate at high wheel speeds continuously. Characterization and mitigation of jitter on a spacecraft is of interest to many missions due to increasingly rigorous attitude stability requirements and the necessity of avoiding excitation of the spacecraft's structural modes. Many instruments require the spacecraft to be held extremely still in order to effectively operate or collect data. Optical instruments in particular often require attitude stability of less than one arcsecond per second in order to avoid optical smear or similar effects.[5,6] Additionally, excessive vibration of a spacecraft may be detrimental to its instruments and operation.

MED induced vibration on a spacecraft is usually characterized through experimentation prior to flight in order to validate requirements. Empirical models are commonly used and allow flywheel imbalance parameters to be extracted.[7,8] In addition to experimental demonstration of jitter performance on an integrated spacecraft, it is of interest to use an analytic model for simulation in the early stages of spacecraft development. A simplified model of flywheel jitter involves including the forces and torques resulting from flywheel imbalance as external disturbances.[3,9–11] This model is well established and is attractive due to its non-computationally expensive formulation – force and torque of jitter are simply proportional to wheel speed squared. Furthermore, the simplified formulation allows a model to be constructed directly from the typical flywheel manufacturer imbalance specifications: static imbalance and dynamic imbalance. The static imbalance force is given by[11]

$$\boldsymbol{F}_{s_i} = U_{s_i}\Omega_i^2\hat{\boldsymbol{u}}_i \tag{1}$$

where U_{s_i} is the static imbalance parameter, Ω_i is wheel speed, and $\hat{\boldsymbol{u}}_i$ is an arbitrary unit vector normal to the wheel spin axis. The dynamic imbalance torque is given by[11]

$$\boldsymbol{L}_{d_i} = U_{d_i}\Omega_i^2\hat{\boldsymbol{v}}_i \tag{2}$$

where U_{d_i} is the dynamic imbalance parameter and $\hat{\boldsymbol{v}}_i$ is an arbitrary unit vector normal to the wheel spin axis. This formulation allows mass imbalances to be implemented as lumped parameters instead of using specific terms such as center of mass location and inertia tensor.[3] Previous literature puts emphasis on empirical modeling of MED jitter and the effects of MED jitter within context of spacecraft flexible dynamics.[12–14] Zhang and Zhang discuss a fully-coupled model of control moment gyro (CMG) imbalance,[15] but present the results without a full derivation and fail to provide the complete system equations of motion.

The simplified "lumped parameter" method of modeling MED jitter is not physically realistic due to the nonconservative nature of adding a system-internal forcing effect as an external disturbance.[16] Since angular momentum is not conserved in this model, a time varying bias in angular velocity is observed. For analysis purposes this does not necessarily present a problem. The overall effect of the angular velocity bias can be quite small for spacecraft that have small wheel imbalance to spacecraft inertia ratios.[17] For spacecraft with poorly balanced reaction wheels or small wheel mass/imbalance to spacecraft inertia ratios this approach may become problematic. Imbalanced CMGs/VSCMGs in particular consist of a stiff differential equation and are not accurately modeled using external disturbances. Additionally, the simplified model does not allow for energy and momentum checks.

If the spacecraft model has other complex behavior such as solar panel flexing or fuel slosh, the importance of energy and momentum validation is of great importance since the coupled nature of these complex spacecraft systems results in extreme difficulty with debugging and validation.

Reference 17 presents a fully-coupled derivation of RW imbalance. It is demonstrated that the fully-coupled model allows an imbalanced RW to be simulated while still using momentum and energy tools for validation of the dynamics. This formulation cannot be used for a CMG/VSCMG however due to the additional degree of freedom in the gimbal.

This paper presents a general method of deriving the equations of motion (EOMs) for a spacecraft containing N VSCMGs/CMGs with imbalanced flywheels. The derivations take a classical mechanics approach, rather than generalized coordinates. The derivation treats the jitter disturbances as true mass imbalances rather than external disturbance forces and torques, and thus represents the true physics governing this fully-coupled phenomenon. As a result, energy and momentum validation tools are available using these models due to the fact that the models obey conservation of angular momentum. Since the spacecraft hub is considered to be rigid, flexible dynamics are not considered. However, the formulation is developed in such a way that adding other modes such as flexing and fuel slosh is relatively straightforward.[18,19] Numerical simulation results of the fully-coupled VSCMG model are provided to demonstrate compliance with conservation of energy/momentum and also to give a direct comparison to the simplified imbalance model.

PROBLEM STATEMENT

The problem consists of modeling static and dynamic imbalance of any number of wheel + gimbal assemblies attached to a rigid spacecraft. In order to develop the equations of motion in a general way, we consider arbitrary locations, inertia tensors, and center of mass locations for the spacecraft hub, gimbal, and wheels. Additionally, the wheel center of mass is not assumed to lie on the gimbal axis of the VSCMG, and the wheel frame origin and gimbal frame origin are not assumed to coincide.

Reference Frame Definitions

The development considers the body frame and N gimbal and wheel frames as well as the inertial frame. The body frame is denoted $\mathcal{B}$. The basis vectors of the body frame are

$$\mathcal{B} : \{B, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\} \tag{3}$$

The i^{th} gimbal and wheel frames are denoted $\mathcal{G}_i$ and $\mathcal{W}_i$, respectively. The basis vectors of $\mathcal{G}_i$ and $\mathcal{W}_i$ are defined as

$$\mathcal{G}_i : \{G_i, \hat{\boldsymbol{g}}_{\text{s}_i}, \hat{\boldsymbol{g}}_{\text{t}_i}, \hat{\boldsymbol{g}}_{\text{g}_i}\} \tag{4}$$

$$\mathcal{W}_i : \{W_i, \hat{\boldsymbol{g}}_{\text{s}_i}, \hat{\boldsymbol{w}}_{2_i}, \hat{\boldsymbol{w}}_{3_i}\} \tag{5}$$

It is assumed that the $\hat{\boldsymbol{g}}_{\text{s}_i}$ vectors of the $\mathcal{G}_i$ and $\mathcal{W}_i$ frames are always parallel.

Variable Definitions

Parameters relating to the spacecraft hub are denoted with a subscript text B. Parameters relating to the The i^{th} gimbal and wheel are denoted with subscripts text G_i and W_i, respectively. The hub, gimbal, and wheel each are allowed center of mass offsets from their respective coordinate frame

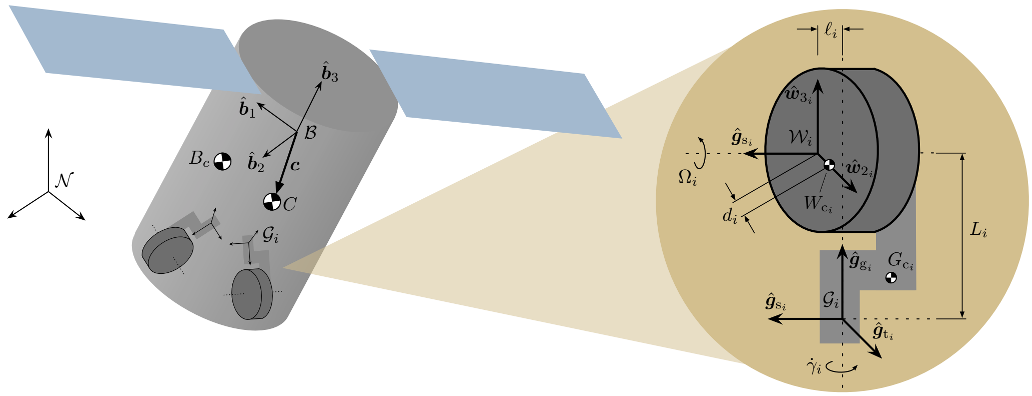


Figure 1. Reference frame setup and variable definitions for the spacecraft + VSCMG problem.

origins. The hub's center of mass location is labeled as B_c. This location is described with respect to the body frame origin as $\boldsymbol{r}_{Bc/B}$. The gimbal is also allowed a general center of mass offset from the gimbal frame origin. This location is labeled as G_{c_i} and is located with respect to the gimbal frame origin as $\boldsymbol{r}_{G_{c_i}/G_i}$. The wheel's center of mass location is labeled somewhat differently. The wheel center of mass is assumed to lie on the $\hat{\boldsymbol{w}}_{2_i}$ axis a length d_i from the wheel frame origin. This does not result in loss of generality since the parameters L_i and ℓ_i describe the axial and transverse offset, respectively, of the wheel origin. Thus, the wheel center of mass location is allowed to vary in three dimensions with respect to the gimbal frame (and thus the body frame as well, since the gimbal origin location does not vary with respect to the body). Since the gimbal and wheel centers of mass change with time, so does the overall spacecraft center of mass. The time-varying center of mass of the entire system is denoted c.

EQUATIONS OF MOTION

The system under consideration is an $2N + 6$ degrees-of-freedom (DOF) system with the following second order terms: inertial acceleration $\ddot{\boldsymbol{r}}_{B/N}$, angular acceleration $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$, the acceleration of each wheel $\dot{\Omega}_1, \ldots, \dot{\Omega}_N$, and the acceleration of the gimbal $\ddot{\gamma}_1, \ldots, \ddot{\gamma}_N$. Thus, a total of $2N + 6$ equations must be developed in order to solve for all second order terms. Section describes the derivation of the translational EOM and represents 3 DOF, section describes the rotational motion and represents 3 DOF, section describes the gimbal torque equation and represents N DOF, and section describes the wheel torque equation and represents N DOF.

Translational Motion

The derivation of the translational EOMs begins with Newton's second law for the center of mass of the spacecraft.

$$\ddot{\boldsymbol{r}}_{C/N} = \frac{\boldsymbol{F}}{m_{\text{sc}}} \tag{6}$$

where

$$m_{\text{sc}} = m_{\text{B}} + \sum_{i=1}^{N} (m_{\text{G}_i} + m_{\text{W}_i}) \tag{7}$$

$\boldsymbol{F}$ is the sum of the external forces on the spacecraft which has mass m_{sc}. Ultimately the acceleration of the body frame or point B is desired, which is expressed through

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \ddot{\boldsymbol{c}} \tag{8}$$

The center of mass $\boldsymbol{c}$ is time variant and is expressed as

$$\boldsymbol{c} = \frac{1}{m_{\text{sc}}}\big(m_{\text{B}}\boldsymbol{r}_{B_c/B} + \sum_{i=1}^{N}(m_{\text{G}_i}\boldsymbol{r}_{G_{c_i}/B} + m_{\text{W}_i}\boldsymbol{r}_{W_{c_i}/B})\big) \tag{9}$$

Find the second inertial derivative of $\boldsymbol{c}$.

$$\dot{\boldsymbol{c}} = \boldsymbol{c}' + \boldsymbol{\omega}\times\boldsymbol{c} \tag{10}$$

$$\ddot{\boldsymbol{c}} = \boldsymbol{c}'' + \dot{\boldsymbol{\omega}}\times\boldsymbol{c} + 2\boldsymbol{\omega}\times\boldsymbol{c}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{c}) \tag{11}$$

The second body frame derivative of the center of mass vector is given by,

$$\begin{aligned}\boldsymbol{c}'' = \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\Bigg[&m_{\text{G}_i}\left(\ddot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}'_{G_{c_i}/B}\right)\\ &+ m_{\text{W}_i}\left(\left(2d_i\dot{\gamma}_i\Omega_i\text{s}\theta_i - d_i\ddot{\gamma}_i\text{c}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\text{s}_i} + \left(\ell_i\ddot{\gamma}_i - d_i\dot{\gamma}_i^2\text{c}\theta_i\right)\hat{\boldsymbol{g}}_{\text{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} + d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i}\right)\Bigg]\end{aligned} \tag{12}$$

Substitute $\ddot{\boldsymbol{c}}$ into Eq. (8).

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \boldsymbol{c}'' + [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{c} \tag{13}$$

Substitute Eq. (12) into Eq. (13) and group second-order terms to obtain the translational equations of motion.

$$\begin{aligned}\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\left[m_{\text{G}_i}[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} - m_{\text{W}_i}d_i\text{c}\theta_i\hat{\boldsymbol{g}}_{\text{s}_i} + m_{\text{W}_i}\ell_i\hat{\boldsymbol{g}}_{\text{t}_i}\right]\ddot{\gamma}_i + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\left[m_{\text{W}_i}d_i\hat{\boldsymbol{w}}_{3_i}\right]\dot{\Omega}_i\\ = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{c} - \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\Big[m_{\text{G}_i}\dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}'_{G_{c_i}/B}\\ + m_{\text{W}_i}\left[\left(2d_i\dot{\gamma}_i\Omega_i\text{s}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\text{s}_i} - d_i\dot{\gamma}_i^2\text{c}\theta_i\hat{\boldsymbol{g}}_{\text{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\right]\Big]\end{aligned} \tag{14}$$

This equation represents 3 DOF and contains all second order states ($\ddot{\boldsymbol{r}}_{B/N}$, $\dot{\boldsymbol{\omega}}$, $\ddot{\gamma}_i$, $\dot{\Omega}_i$). Removing wheel imbalance terms and assuming a symmetrical VSCMG (i.e. $\boldsymbol{r}_{G_{c_i}/G_i} = \boldsymbol{0}$, $\ell_i = 0$, $d_i = 0$) gives the following equation.

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} = \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c} \tag{15}$$

Thus, the balanced VSCMG translational equation of motion does not contain any second-order terms relating to the wheel or gimbal, and agrees with Reference.[16] The following section shows the derivation of the rotational equations of motion.

Rotational Motion

The derivation of rotational EOMs starts with the angular momentum of the spacecraft about point B.

$$\boldsymbol{H}_{\text{sc},B} = \boldsymbol{H}_{\text{B},B} + \sum_{i=1}^{N}(\boldsymbol{H}_{\text{G}_i,B} + \boldsymbol{H}_{\text{W}_i,B}) \tag{16}$$

The inertial time derivative of angular momentum when the body fixed coordinate frame origin is not coincident with the center of mass of the body is

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{L}_B + m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} \tag{17}$$

where $\boldsymbol{L}_B$ is the vector sum of external torques acting on the spacecraft. Differentiating Eq. (16), the inertial derivative of the spacecraft angular momentum is expressed as

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \dot{\boldsymbol{H}}_{\text{B},B} + \sum_{i=1}^{N}(\dot{\boldsymbol{H}}_{\text{G}_i,B} + \dot{\boldsymbol{H}}_{\text{W}_i,B}) \tag{18}$$

Thus, in order to use Eq. (17), each derivative on the right-hand side of Eq. (18) needs to be evaluated. The first step is to derive the hub angular momentum derivative $\dot{\boldsymbol{H}}_{\text{B},B}$. The hub angular momentum about point B_c is given by

$$\boldsymbol{H}_{\text{B},B_c} = [I_{\text{B},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{19}$$

Angular momentum about point B_c is related to point B using the following equation.

$$\boldsymbol{H}_{\text{B},B} = \boldsymbol{H}_{\text{B},B_c} + m_{\text{B}}\boldsymbol{r}_{B_c/B} \times \dot{\boldsymbol{r}}_{B_c/B} \tag{20}$$

Taking the inertial time derivative of the hub's angular momentum yields

$$\dot{\boldsymbol{H}}_{\text{B},B} = [I_{\text{B},B_c}]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I_{\text{B},B_c}]\boldsymbol{\omega} + m_{\text{B}}\boldsymbol{r}_{B_c/B} \times \ddot{\boldsymbol{r}}_{B_c/B} \tag{21}$$

Note that the body rate pseudovector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ will be abbreviated as $\boldsymbol{\omega}$ henceforth. Knowing that $\boldsymbol{r}_{B_c/B}$ is fixed with respect to the body frame, the following my be defined

$$\ddot{\boldsymbol{r}}_{B_c/B} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{B_c/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{B_c/B}) \tag{22}$$

Substitute Eq. (22) into Eq. (21) yields

$$\dot{\boldsymbol{H}}_{\text{B},B} = [I_{\text{B},B_c}]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I_{\text{B},B_c}]\boldsymbol{\omega} + m_{\text{B}}\boldsymbol{r}_{B_c/B} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{B_c/B}) + m_{\text{B}}\boldsymbol{r}_{B_c/B} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{B_c/B})) \tag{23}$$

Employing the Jacobi triple-product identity, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} + \boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c})$, on the right-hand side of Eq. (23)

$$\dot{\boldsymbol{H}}_{\text{B},B} = [I_{\text{B},B_c}]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I_{\text{B},B_c}]\boldsymbol{\omega} + m_{\text{B}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\dot{\boldsymbol{\omega}} + m_{\text{B}}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\boldsymbol{\omega} \tag{24}$$

The parallel axis theorem relates inertia about the hub center of B_c to the hub origin B.

$$[I_{\text{B},B}] = [I_{\text{B},B_c}] + m_{\text{B}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T \tag{25}$$

The hub angular momentum derivative simplifies to

$$\dot{\boldsymbol{H}}_{\mathrm{B},B} = [I_{\mathrm{B},B}]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{B},B}]\boldsymbol{\omega} \tag{26}$$

The next step is to derive the gimbal angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{G}_i,B}$. The angular velocity of the gimbal frame with respect to inertial is

$$\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{B}} = \boldsymbol{\omega} + \dot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} \tag{27}$$

The gimbal angular momentum about point G_{c_i} is given by

$$\boldsymbol{H}_{\mathrm{G}_i,G_{c_i}} = [I_{\mathrm{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} = [I_{\mathrm{G}_i,G_{c_i}}](\boldsymbol{\omega} + \dot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i}) \tag{28}$$

Angular momentum about point G_{c_i} is related to point B using the following equation.

$$\boldsymbol{H}_{\mathrm{G}_i,B} = \boldsymbol{H}_{\mathrm{G}_i,G_{c_i}} + m_{\mathrm{G}_i}\boldsymbol{r}_{G_{c_i}/B} \times \dot{\boldsymbol{r}}_{G_{c_i}/B} \tag{29}$$

Take the inertial derivative.

$$\dot{\boldsymbol{H}}_{\mathrm{G}_i,B} = [I_{\mathrm{G}_i,G_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i}) + [I_{\mathrm{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + m_{\mathrm{G}_i}\boldsymbol{r}_{G_{c_i}/B} \times \ddot{\boldsymbol{r}}_{G_{c_i}/B} \tag{30}$$

The next step is to define the gimbal inertia tensor about the gimbal center of mass $[I_{\mathrm{G}_i,G_{c_i}}]$ and its body frame derivative $[I_{\mathrm{G}_i,G_{c_i}}]'$. Expressed in the gimbal frame,

$$[I_{\mathrm{G}_i,G_{c_i}}] = {}^{\mathcal{G}_i}\begin{bmatrix} I_{\mathrm{G}_{s_i}} & I_{\mathrm{G}_{12_i}} & I_{\mathrm{G}_{13_i}} \\ I_{\mathrm{G}_{12_i}} & I_{\mathrm{G}_{t_i}} & I_{\mathrm{G}_{23_i}} \\ I_{\mathrm{G}_{13_i}} & I_{\mathrm{G}_{23_i}} & I_{\mathrm{G}_{g_i}} \end{bmatrix} \tag{31}$$

By expressing this tensor in a frame independent form, the body frame derivative is found to be,

$$[I_{\mathrm{G}_i,G_{c_i}}]' = \dot{\gamma}_i \; {}^{\mathcal{G}_i}\begin{bmatrix} -2I_{\mathrm{G}_{12_i}} & (I_{\mathrm{G}_{\mathrm{s}_i}} - I_{\mathrm{G}_{\mathrm{t}_i}}) & -I_{\mathrm{G}_{23_i}} \\ (I_{\mathrm{G}_{\mathrm{s}_i}} - I_{\mathrm{G}_{\mathrm{t}_i}}) & 2I_{\mathrm{G}_{12_i}} & I_{\mathrm{G}_{13_i}} \\ -I_{\mathrm{G}_{23_i}} & I_{\mathrm{G}_{13_i}} & 0 \end{bmatrix} \tag{32}$$

The second inertial derivative of $\boldsymbol{r}_{G_{c_i}/B}$ is needed. Define the body frame derivative and first inertial derivative of $\boldsymbol{r}_{G_{c_i}/B}$, noting that point G_i is fixed with respect to point B.

$$\boldsymbol{r}_{G_{c_i}/B} = \boldsymbol{r}_{G_{c_i}/G_i} + \boldsymbol{r}_{G_i/B} \tag{33}$$

$$\boldsymbol{r}'_{G_{c_i}/B} = \boldsymbol{r}'_{G_{c_i}/G_i} = \dot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} \times \boldsymbol{r}_{G_{c_i}/G_i} \tag{34}$$

$$\dot{\boldsymbol{r}}_{G_{c_i}/B} = \boldsymbol{r}'_{G_{c_i}/B} + \boldsymbol{\omega} \times \boldsymbol{r}_{G_{c_i}/B} = \dot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} \times \boldsymbol{r}_{G_{c_i}/G_i} + \boldsymbol{\omega} \times \boldsymbol{r}_{G_{c_i}/B} \tag{35}$$

The second inertial derivative of $\boldsymbol{r}_{G_{c_i}/B}$ is

$$\begin{aligned} \ddot{\boldsymbol{r}}_{G_{c_i}/B} =& \ddot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} \times \boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} \times \boldsymbol{r}'_{G_{c_i}/G_i} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{c_i}/B} + \boldsymbol{\omega} \times \boldsymbol{r}'_{G_{c_i}/B} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{G_{c_i}/B} \\ =& \ddot{\gamma}_i [\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i [\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [\tilde{\boldsymbol{r}}_{G_{c_i}/B}]^T \dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{G_{c_i}/B}]^T \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{c_i}/B} \end{aligned} \tag{36}$$

Note that ,

$$\boldsymbol{r}''_{G_{c_i}/B} = \boldsymbol{r}''_{G_{c_i}/G} = \ddot{\gamma}_i [\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i [\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} \tag{37}$$

Substitute into Eq. (30).

$$\dot{\boldsymbol{H}}_{\mathrm{G}_i,B} = [I_{\mathrm{G}_i,G_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i}) + [I_{\mathrm{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} \\ + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\left[\ddot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [\tilde{\boldsymbol{r}}_{G_{c_i}/B}]^T\dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{G_{c_i}/B}]^T\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{G_{c_i}/B}\right] \tag{38}$$

The parallel axis theorem relating the gimbal inertia about point B to the gimbal inertia about point G_{c_i} is given by

$$[I_{\mathrm{G}_i,B}] = [I_{\mathrm{G}_i,G_{c_i}}] + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]^T \tag{39}$$

Using Eq. (39), Eq. (38) simplifies to

$$\dot{\boldsymbol{H}}_{\mathrm{G}_i,B} = [I_{\mathrm{G}_i,B}]\dot{\boldsymbol{\omega}} + [I_{\mathrm{G}_i,G_{c_i}}]\ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + [I_{\mathrm{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,B}]\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} \\ + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\left[\ddot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + 2[\tilde{\boldsymbol{r}}'_{G_{c_i}/B}]^T\boldsymbol{\omega}\right] \tag{40}$$

The next step is to employ the body frame derivative of the parallel axis theorem.

$$[I_{\mathrm{G}_i,B}]' = [I_{\mathrm{G}_i,G_{c_i}}]' + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}'_{G_{c_i}/B}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]^T + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\boldsymbol{r}}'_{G_{c_i}/B}]^T \tag{41}$$

Eq. (40) is further simplified using Eq. (41) to give the gimbal angular momentum derivative.

$$\dot{\boldsymbol{H}}_{\mathrm{G}_i,B} = [I_{\mathrm{G}_i,B}]\dot{\boldsymbol{\omega}} + [I_{\mathrm{G}_i,B}]'\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,B}]\boldsymbol{\omega} + [I_{\mathrm{G}_i,G_{c_i}}]\ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + [I_{\mathrm{G}_i,G_{c_i}}]'\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} \\ + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\left[\ddot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + \dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i}\right] + m_{\mathrm{G}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\boldsymbol{r}'_{G_{c_i}/B} \tag{42}$$

The next step is to derive the wheel angular momentum derivative $\dot{\boldsymbol{H}}_{\mathrm{W}_i,B}$. The angular velocity of the wheel with respect to inertial is

$$\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{G}_i} = \boldsymbol{\omega} + \dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \Omega_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} \tag{43}$$

The wheel angular momentum about point W_{c_i} is given by

$$\boldsymbol{H}_{\mathrm{W}_i,W_{c_i}} = [I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} = [I_{\mathrm{W}_i,W_{c_i}}](\boldsymbol{\omega} + \dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \Omega_i\hat{\boldsymbol{g}}_{\mathrm{s}_i}) \tag{44}$$

Angular momentum about point W_{c_i} is related to point B using the following equation.

$$\boldsymbol{H}_{\mathrm{W}_i,B} = \boldsymbol{H}_{\mathrm{W}_i,W_{c_i}} + m_{\mathrm{W}_i}\boldsymbol{r}_{W_{c_i}/B} \times \dot{\boldsymbol{r}}_{W_{c_i}/B} \tag{45}$$

Take the inertial derivative.

$$\dot{\boldsymbol{H}}_{\mathrm{W}_i,B} = [I_{\mathrm{W}_i,W_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \\ + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathrm{W}_i}\boldsymbol{r}_{W_{c_i}/B} \times \ddot{\boldsymbol{r}}_{W_{c_i}/B} \tag{46}$$

The body relative inertia tensor time derivative $[I_{\mathrm{rw}_i,W_{c_i}}]'$ needs to be defined. For this general RW model, the inertia matrix of the RW in the $\mathcal{W}_i$ frame is defined as

$$[I_{\mathrm{W}_i,W_{c_i}}] = {}^{\mathcal{W}_i}\begin{bmatrix} J_{11_i} & J_{12_i} & J_{13_i} \\ J_{12_i} & J_{22_i} & J_{23_i} \\ J_{13_i} & J_{23_i} & J_{33_i} \end{bmatrix} \tag{47}$$

The definition of $[I_{\mathrm{W}_i,W_{c_i}}]$ allows for any RW inertia matrix to be considered. In order to take the body frame derivative of $[I_{\mathrm{W}_i,W_{c_i}}]$, Eq. (47) is rewritten in a general form using outer product expansions. After an exhausting amount of algebra, the body frame derivative of Eq. (47) simplifies to the following tensor (given in wheel frame components).

$$[I_{\mathrm{W}_i,W_{c_i}}]' = \\ {}^{\mathcal{W}_i}\begin{bmatrix} 2\dot{\gamma}_i(J_{13_i}\mathrm{s}\theta_i - J_{12_i}\mathrm{c}\theta_i) & \dot{\gamma}_i(J_{a_i}\mathrm{c}\theta_i + J_{23_i}\mathrm{s}\theta_i) - J_{13_i}\Omega_i & \dot{\gamma}_i(J_{b_i}\mathrm{s}\theta_i - J_{23_i}\mathrm{c}\theta_i) + J_{12_i}\Omega_i \\ \dot{\gamma}_i(J_{a_i}\mathrm{c}\theta_i + J_{23_i}\mathrm{s}\theta_i) - J_{13_i}\Omega_i & 2(J_{12_i}\dot{\gamma}_i\mathrm{c}\theta_i - J_{23_i}\Omega_i) & \dot{\gamma}_i(J_{13_i}\mathrm{c}\theta_i - J_{12_i}\mathrm{s}\theta_i) + J_{c_i}\Omega_i \\ \dot{\gamma}_i(J_{b_i}\mathrm{s}\theta_i - J_{23_i}\mathrm{c}\theta_i) + J_{12_i}\Omega_i & \dot{\gamma}_i(J_{13_i}\mathrm{c}\theta_i - J_{12_i}\mathrm{s}\theta_i) + J_{c_i}\Omega_i & 2(J_{23_i}\Omega_i - J_{13_i}\dot{\gamma}_i\mathrm{s}\theta_i) \end{bmatrix} \tag{48}$$

Where,

$$J_{a_i} = J_{11_i} - J_{22_i} \tag{49}$$
$$J_{b_i} = J_{33_i} - J_{11_i} \tag{50}$$
$$J_{c_i} = J_{22_i} - J_{33_i} \tag{51}$$

The second inertial derivative of $\boldsymbol{r}_{W_{c_i}/B}$ is needed. Note that the static imbalance is fundamentally an impact of the wheel center of mass offset d_i. We arbitrarily allow this offset to act in the $\hat{\boldsymbol{w}}_{2_i}$ direction. The center of mass of the wheel with respect to point W_i is thus given by

$$\boldsymbol{r}_{W_{c_i}/W_i} = d_i\hat{\boldsymbol{w}}_{2_i} \tag{52}$$

Additionally, point W does not lie on the body fixed gimbal axis $\hat{\boldsymbol{g}}_{\mathrm{g}_i}$ for all VSCMGs. Such an offset subtly contributes to jitter. Thus, we introduce a radial offset ℓ_i of the wheel center of mass. Point W_i is related to point G_i by

$$\boldsymbol{r}_{W_i/G_i} = \ell_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + L_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} \tag{53}$$

where L_i is the axial offset of the wheel from the gimbal origin that is common in many VSCMGs. The time varying vector that relates the wheel center of mass to the body frame origin is then given by

$$\boldsymbol{r}_{W_{c_i}/B} = \boldsymbol{r}_{W_{c_i}/W_i} + \boldsymbol{r}_{W_i/G_i} + \boldsymbol{r}_{G_i/B} = d_i\hat{\boldsymbol{w}}_{2_i} + \ell_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + L_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \boldsymbol{r}_{G_i/B} \tag{54}$$

Recalling that $\boldsymbol{r}_{G_i/B}$ and $\hat{\boldsymbol{g}}_{\mathrm{g}_i}$ are both body frame fixed vectors, we define the body frame derivative and first inertial derivative of $\boldsymbol{r}_{W_{c_i}/B}$.

$$\boldsymbol{r}'_{W_{c_i}/B} = \boldsymbol{r}'_{W_{c_i}/G} = d_i\hat{\boldsymbol{w}}'_{2_i} + \ell_i\hat{\boldsymbol{g}}'_{\mathrm{s}_i} = d_i\Omega_i\hat{\boldsymbol{w}}_{3_i} - d_i\dot{\gamma}_i\mathrm{c}\theta_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \ell_i\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{t}_i} \tag{55}$$
$$\dot{\boldsymbol{r}}_{W_{c_i}/B} = \boldsymbol{r}'_{W_{c_i}/B} + \boldsymbol{\omega}\times\boldsymbol{r}_{W_{c_i}/B} \tag{56}$$

The second inertial derivative of $\boldsymbol{r}_{W_{c_i}/B}$ is

$$\begin{aligned} \ddot{\boldsymbol{r}}_{W_{c_i}/B} =& \boldsymbol{r}''_{W_{c_i}/B} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{W_{c_i}/B} + \boldsymbol{\omega}\times\boldsymbol{r}'_{W_{c_i}/B} + \boldsymbol{\omega}\times\dot{\boldsymbol{r}}_{W_{c_i}/B} \\ =& \boldsymbol{r}''_{W_{c_i}/B} + [\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T\dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{W_{c_i}/B} \end{aligned} \tag{57}$$

The second body frame derivative of $\boldsymbol{r}_{W_{c_i}/B}$ is given by,

$$\boldsymbol{r}''_{W_{c_i}/B} = \left(2d_i\dot{\gamma}_i\Omega_i\mathrm{s}\theta_i - d_i\ddot{\gamma}_i\mathrm{c}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \left(\ell_i\ddot{\gamma}_i - d_i\dot{\gamma}_i^2\mathrm{c}\theta_i\right)\hat{\boldsymbol{g}}_{\mathrm{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} + d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} \tag{58}$$

Substitute Eq. (57) into Eq. (46).

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{W}_i,B} =&[I_{\mathrm{W}_i,W_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \\ &+ m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\left[\boldsymbol{r}''_{W_{c_i}/B} + [\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T\dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{r}_{W_{c_i}/B}\right]\end{aligned} \tag{59}$$

The parallel axis theorem relating the gimbal inertia about point B to the gimbal inertia about point W_{c_i} is given by

$$[I_{\mathrm{W}_i,B}] = [I_{\mathrm{W}_i,W_{c_i}}] + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T \tag{60}$$

Using Eq. (60), Eq. (59) simplifies to

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{W}_i,B} =&[I_{\mathrm{W}_i,B}]\dot{\boldsymbol{\omega}} + [I_{\mathrm{W}_i,W_{c_i}}](\ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,B}]\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \\ &+ [I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\left[\boldsymbol{r}''_{W_{c_i}/B} + 2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T\boldsymbol{\omega}\right]\end{aligned} \tag{61}$$

The next step is to employ the body frame derivative of the parallel axis theorem.

$$[I_{\mathrm{W}_i,B}]' = [I_{\mathrm{W}_i,W_{c_i}}]' + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}][\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \tag{62}$$

Eq. (61) is further simplified using Eq. (62) to give the wheel angular momentum derivative.

$$\begin{aligned}\dot{\boldsymbol{H}}_{\mathrm{W}_i,B} =&[I_{\mathrm{W}_i,B}]\dot{\boldsymbol{\omega}} + [I_{\mathrm{W}_i,B}]'\boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,B}]\boldsymbol{\omega} + [I_{\mathrm{W}_i,W_{c_i}}](\ddot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \\ &+ [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}''_{W_{c_i}/B} + m_{\mathrm{W}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\end{aligned} \tag{63}$$

We may now formulate the rotational equations of motion. Euler's equation is rearranged as

$$m_{\mathrm{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{H}}_{\mathrm{B},B} + \sum_{i=1}^{N}(\dot{\boldsymbol{H}}_{\mathrm{G}_i,B} + \dot{\boldsymbol{H}}_{\mathrm{W}_i,B}) = \boldsymbol{L}_B \tag{64}$$

The rotational equations of motion are formulated by substituting Equations (26), (42), and (63) into Eq. (64)

$$\begin{aligned}&m_{\mathrm{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\mathrm{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N}\Big[[I_{\mathrm{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{g}_i} + m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + [I_{\mathrm{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{g}_i} \\ &\quad + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}](\ell_i\hat{\boldsymbol{g}}_{\mathrm{t}_i} - d_i\mathrm{c}\theta_i\hat{\boldsymbol{g}}_{\mathrm{s}_i})\Big]\ddot{\gamma}_i + \sum_{i=1}^{N}\Big[[I_{\mathrm{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{s}_i} + m_{\mathrm{W}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big]\dot{\Omega}_i \\ &= \boldsymbol{L}_B - [I_{\mathrm{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\mathrm{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[[I_{\mathrm{G}_i,G_{c_i}}]'\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathrm{g}_i} + m_{\mathrm{G}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\boldsymbol{r}'_{G_{c_i}/B} \\ &\quad + m_{\mathrm{G}_i}\dot{\gamma}_i[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [I_{\mathrm{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i} + [I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \\ &\quad + m_{\mathrm{W}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B} + m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\left[\left(2d_i\dot{\gamma}_i\Omega_i\mathrm{s}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\mathrm{s}_i} - d_i\dot{\gamma}_i^2\mathrm{c}\theta_i\hat{\boldsymbol{g}}_{\mathrm{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\right]\Big]\end{aligned} \tag{65}$$

The total spacecraft inertia about point B is given by,

$$[I_{\text{sc},B}] = [I_{\text{B},B}] + \sum_{i=1}^{N}[I_{\text{vscmg}_i,B}] \tag{66}$$

where,

$$[I_{\text{vscmg}_i,B}] = [I_{\text{G}_i,B}] + [I_{\text{W}_i,B}] \tag{67}$$

The rotational equation of motion for a VSCMG with balanced wheels may be found by setting imbalance terms to zero.

$$\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N} I_{\text{V}_{g_i}}\hat{\boldsymbol{g}}_{\text{g}_i}\ddot{\gamma}_i + \sum_{i=1}^{N} I_{\text{W}_{s_i}}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i& \\
= \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[\omega_t\dot{\gamma}_i(I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}} + I_{\text{V}_{g_i}})\hat{\boldsymbol{g}}_{\text{s}_i}& \\
+ \big[\omega_s\dot{\gamma}_i(I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}} - I_{\text{V}_{g_i}}) + I_{\text{W}_{s_i}}\Omega_i(\dot{\gamma} + \omega_g)\big]\hat{\boldsymbol{g}}_{\text{t}_i} - \omega_t I_{\text{W}_{s_i}}\Omega_i\hat{\boldsymbol{g}}_{\text{g}_i}\Big]&
\end{aligned} \tag{68}$$

Gimbal Torque Equation

The gimbal torque equation is used to relate body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and gimbal rate derivative $\ddot{\gamma}_i$. The VSCMG motor torque u_{g_i} is the $\hat{\boldsymbol{g}}_{\text{g}_i}$ component of gimbal torque about point G_i. The torque acting on a VSCMG at the joint between the motor and the gimbal assembly is given by

$$\boldsymbol{L}_{G_i} = {}^{\mathcal{G}_i}\!\begin{bmatrix}\tau_{g_{\text{s}_i}} \\ \tau_{g_{\text{t}_i}} \\ u_{\text{g}_i}\end{bmatrix} \tag{69}$$

The transverse torques acting on the gimbal $\tau_{g_{\text{s}_i}}$ and $\tau_{g_{\text{t}_i}}$ are structural torques and do not contribute to the equation. Torque about point G_i is related to torque about the VSCMG center of mass V_{c_i} using the following equation.

$$\boldsymbol{L}_{G_i} = \boldsymbol{L}_{V_{c_i}} + m_{\text{V}_i}\boldsymbol{r}_{V_{c_i}/G_i} \times \ddot{\boldsymbol{r}}_{V_{c_i}/N} \tag{70}$$

Euler's equation applies as follows.

$$\boldsymbol{L}_{V_{c_i}} = \dot{\boldsymbol{H}}_{\text{G}_i,V_{c_i}} + \dot{\boldsymbol{H}}_{\text{W}_i,V_{c_i}} \tag{71}$$

The VSCMG motor torque is the $\hat{\boldsymbol{g}}_{\text{g}_i}$ component of the right-hand side of Eq. (70). This is found in a frame independent format as

$$u_{\text{g}_i} = \hat{\boldsymbol{g}}_{\text{g}_i}^T\boldsymbol{L}_{G_i} = \hat{\boldsymbol{g}}_{\text{g}_i}^T\big(\dot{\boldsymbol{H}}_{\text{G}_i,V_{c_i}} + \dot{\boldsymbol{H}}_{\text{W}_i,V_{c_i}} + \boldsymbol{r}_{V_{c_i}/G_i} \times m_{\text{V}_i}\ddot{\boldsymbol{r}}_{V_{c_i}/N}\big) \tag{72}$$

where the gimbal and wheel angular momentum derivatives about point V_{c_i} are related to point W_{c_i} using the following equation.

$$\dot{\boldsymbol{H}}_{\text{G}_i,V_{c_i}} = \dot{\boldsymbol{H}}_{\text{G}_i,G_{c_i}} + m_{\text{G}_i}\boldsymbol{r}_{G_{c_i}/V_{c_i}} \times \ddot{\boldsymbol{r}}_{G_{c_i}/V_{c_i}} \tag{73}$$

$$\dot{\boldsymbol{H}}_{\text{W}_i,V_{c_i}} = \dot{\boldsymbol{H}}_{\text{W}_i,W_{c_i}} + m_{\text{W}_i}\boldsymbol{r}_{W_{c_i}/V_{c_i}} \times \ddot{\boldsymbol{r}}_{W_{c_i}/V_{c_i}} \tag{74}$$

The inertial derivatives of the wheel and gimbal angular momentum about their respective centers of mass were found in the previous section and are reprinted here for the reader's convenience.

$$\dot{\boldsymbol{H}}_{\mathrm{G}_i,G_{c_i}} = [I_{\mathrm{G}_i,G_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i}) + [I_{\mathrm{G}_i,G_{c_i}}]' \boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} \tag{75}$$

$$\dot{\boldsymbol{H}}_{\mathrm{W}_i,W_{c_i}} = [I_{\mathrm{W}_i,W_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i \hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega \dot{\gamma} \hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [I_{\mathrm{W}_i,W_{c_i}}]' \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \tag{76}$$

Define the terms on the right-hand side of Eq. (73).

$$\ddot{\boldsymbol{r}}_{G_{c_i}/V_{c_i}} = \boldsymbol{r}''_{G_{c_i}/V_{c_i}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{G_{c_i}/V_{c_i}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{G_{c_i}/V_{c_i}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{G_{c_i}/V_{c_i}}) \tag{77}$$

Define the terms on the right-hand side of Eq. (74).

$$\ddot{\boldsymbol{r}}_{W_{c_i}/V_{c_i}} = \boldsymbol{r}''_{W_{c_i}/V_{c_i}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{W_{c_i}/V_{c_i}} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{W_{c_i}/V_{c_i}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{W_{c_i}/V_{c_i}}) \tag{78}$$

The VSCMG center of mass location with respect to point G_i and its body frame derivatives are given by

$$\boldsymbol{r}_{V_{c_i}/G_i} = \frac{1}{m_{\mathrm{V}_i}} \Big(m_{\mathrm{G}_i} \boldsymbol{r}_{G_{c_i}/G_i} + m_{\mathrm{W}_i} \boldsymbol{r}_{W_{c_i}/G_i} \Big) \tag{79}$$

$$\boldsymbol{r}'_{V_{c_i}/G_i} = \rho_{G_i} \boldsymbol{r}'_{G_{c_i}/G_i} + \rho_{W_i} \boldsymbol{r}'_{W_{c_i}/G_i} \tag{80}$$

$$\boldsymbol{r}''_{V_{c_i}/G_i} = \rho_{G_i} \boldsymbol{r}''_{G_{c_i}/G_i} + \rho_{W_i} \boldsymbol{r}''_{W_{c_i}/G_i} \tag{81}$$

where the mass ratios are abbreviated as

$$\rho_{G_i} = \frac{m_{\mathrm{G}_i}}{m_{\mathrm{G}_i} + m_{\mathrm{W}_i}} \tag{82}$$

$$\rho_{W_i} = \frac{m_{\mathrm{W}_i}}{m_{\mathrm{G}_i} + m_{\mathrm{W}_i}} \tag{83}$$

$\ddot{\boldsymbol{r}}_{V_{c_i}/N}$ is expanded as,

$$\ddot{\boldsymbol{r}}_{V_{c_i}/N} = \ddot{\boldsymbol{r}}_{V_{c_i}/B} + \ddot{\boldsymbol{r}}_{B/N} \tag{84}$$

Find the second inertial derivative of $\boldsymbol{r}_{V_{c_i}/B}$ (note that $\boldsymbol{r}'_{V_{c_i}/G} = \boldsymbol{r}'_{V_{c_i}/B}$ and $\boldsymbol{r}''_{V_{c_i}/G} = \boldsymbol{r}''_{V_{c_i}/B}$)

$$\dot{\boldsymbol{r}}_{V_{c_i}/B} = \boldsymbol{r}'_{V_{c_i}/B} + \boldsymbol{\omega} \times \boldsymbol{r}_{V_{c_i}/B} \tag{85}$$

$$\ddot{\boldsymbol{r}}_{V_{c_i}/B} = \boldsymbol{r}''_{V_{c_i}/B} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{V_{c_i}/B} + 2\boldsymbol{\omega} \times \boldsymbol{r}'_{V_{c_i}/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{V_{c_i}/B}) \tag{86}$$

Substituting into Eq. (72) and performing a massive rearrange gives the VSCMG gimbal torque equation of motion.

$$\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\Big[m_{\mathrm{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\Big]\ddot{\boldsymbol{r}}_{B/N}+\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\Big[[I_{\mathrm{V}_i,V_{c_i}}]+m_{\mathrm{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}][\tilde{\boldsymbol{r}}_{V_{c_i}/B}]^T\Big]\dot{\boldsymbol{\omega}}+\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\Big[[I_{\mathrm{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{g}_i}$$
$$+[I_{\mathrm{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{g}_i}+[P_i]\big(\ell_i\hat{\boldsymbol{g}}_{\mathrm{t}_i}-d_i\mathrm{c}\theta_i\hat{\boldsymbol{g}}_{\mathrm{s}_i}\big)+[Q_i][\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i}\Big]\ddot{\gamma}_i+\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\Big[[I_{\mathrm{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathrm{s}_i}+[P_i]d_i\hat{\boldsymbol{w}}_{3_i}\Big]\dot{\Omega}_i$$
$$=-\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\Big[\dot{\gamma}_i[Q_i][\tilde{\hat{\boldsymbol{g}}}_{\mathrm{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i}+[P_i]\big[\big(2d_i\dot{\gamma}_i\Omega_i\mathrm{s}\theta_i-\ell_i\dot{\gamma}_i^2\big)\hat{\boldsymbol{g}}_{\mathrm{s}_i}-d_i\dot{\gamma}_i^2\mathrm{c}\theta_i\hat{\boldsymbol{g}}_{\mathrm{t}_i}-d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\big]$$
$$+[I_{\mathrm{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}}+[\tilde{\boldsymbol{\omega}}][I_{\mathrm{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}}+[I_{\mathrm{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathrm{t}_i}+[I_{\mathrm{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}}$$
$$+[\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}}+m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{G_{c_i}/V_{c_i}}+[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{G_{c_i}/V_{c_i}}\big)$$
$$+m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{W_{c_i}/V_{c_i}}+[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{W_{c_i}/V_{c_i}}\big)+m_{\mathrm{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{V_{c_i}/B}+[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{V_{c_i}/B}\big)\Big]+u_{\mathrm{g}_i} \tag{87}$$

Where,

$$[I_{\mathrm{V}_i,V_{c_i}}]=[I_{\mathrm{G}_i,V_{c_i}}]+[I_{\mathrm{W}_i,V_{c_i}}] \tag{88}$$
$$[I_{\mathrm{G}_i,V_{c_i}}]=[I_{\mathrm{G}_i,G_{c_i}}]+m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]^T \tag{89}$$
$$[I_{\mathrm{W}_i,V_{c_i}}]=[I_{\mathrm{W}_i,W_{c_i}}]+m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]^T \tag{90}$$
$$[P_i]=m_{\mathrm{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]-m_{\mathrm{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]+m_{\mathrm{W}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{91}$$
$$[Q_i]=m_{\mathrm{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]-m_{\mathrm{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]+m_{\mathrm{G}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{92}$$
$$[\tilde{\boldsymbol{\omega}}]^2=[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \tag{93}$$

Removing all imbalance terms, Eq. (87) simplifies to the equation found in Reference.[16]

$$I_{\mathrm{V}_{g_i}}(\hat{\boldsymbol{g}}_{\mathrm{g}_i}^T\dot{\boldsymbol{\omega}}+\ddot{\gamma}_i)=u_{\mathrm{g}_i}+(I_{\mathrm{V}_{s_i}}-I_{\mathrm{V}_{t_i}})\omega_s\omega_t+I_{\mathrm{W}_{s_i}}\Omega_i\omega_t \tag{94}$$

Wheel Torque Equation

The wheel torque equation is used to relate body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and wheel speed derivative $\dot{\Omega}_i$. The wheel motor torque u_{s_i} is the $\hat{\boldsymbol{g}}_{\mathrm{s}_i}$ component of wheel torque about point W_i. The torque acting on a RW at the joint between the RW motor and the RW rotor is given by

$$\boldsymbol{L}_{W_i}={}^{\mathcal{W}_i}\begin{bmatrix}u_{\mathrm{s}_i}\\ \tau_{w_{2_i}}\\ \tau_{w_{3_i}}\end{bmatrix} \tag{95}$$

The transverse torques acting on the gimbal $\tau_{w_{2_i}}$ and $\tau_{w_{3_i}}$ are structural torques and do not contribute to the equation. Torque about point W_i is related to torque about W_{c_i} using the following equation.

$$\boldsymbol{L}_{W_i}=\boldsymbol{L}_{W_{c_i}}+\boldsymbol{r}_{W_{c_i}/W_i}\times m_{\mathrm{W}_i}\ddot{\boldsymbol{r}}_{W_{c_i}/N} \tag{96}$$

Euler's equation applied as follows.

$$\boldsymbol{L}_{W_{c_i}} = \dot{\boldsymbol{H}}_{\mathrm{W}_i, W_{c_i}} \tag{97}$$

The VSCMG motor torque is the $\hat{\boldsymbol{g}}_{\mathrm{g}_i}$ component of the right-hand side of Eq. (96). This is found in a frame independent format as

$$u_{\mathrm{s}_i} = \hat{\boldsymbol{g}}_{\mathrm{s}_i}^T \boldsymbol{L}_{W_i} = \hat{\boldsymbol{g}}_{\mathrm{s}_i}^T \big(\dot{\boldsymbol{H}}_{\mathrm{W}_i, W_{c_i}} + \boldsymbol{r}_{W_{c_i}/W_i} \times m_{\mathrm{W}_i} \ddot{\boldsymbol{r}}_{W_{c_i}/N} \big) \tag{98}$$

The inertial derivatives of the wheel and gimbal angular momentum about their respective centers of mass were found in the previous section and are reprinted here for the reader's convenience.

$$\dot{\boldsymbol{H}}_{\mathrm{W}_i, W_{c_i}} = [I_{\mathrm{W}_i, W_{c_i}}](\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i \hat{\boldsymbol{g}}_{\mathrm{g}_i} + \dot{\Omega}_i \hat{\boldsymbol{g}}_{\mathrm{s}_i} + \Omega \dot{\gamma} \hat{\boldsymbol{g}}_{\mathrm{t}_i}) + [I_{\mathrm{W}_i, W_{c_i}}]' \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i, W_{c_i}}] \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \tag{99}$$

Define

$$\ddot{\boldsymbol{r}}_{W_{c_i}/N} = \boldsymbol{r}''_{W_{c_i}/B} + [\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T \dot{\boldsymbol{\omega}} + 2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \boldsymbol{r}_{W_{c_i}/B} + \ddot{\boldsymbol{r}}_{B/N} \tag{100}$$

The second body frame derivative of $\boldsymbol{r}_{W_{c_i}/B}$ was defined in Eq. (58). Substituting into Eq. (98) gives the wheel torque equation.

$$\begin{aligned}
&\left[m_{\mathrm{W}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T \right] \ddot{\boldsymbol{r}}_{B/N} + \left[\hat{\boldsymbol{g}}_{\mathrm{s}_i}^T [I_{\mathrm{W}_i, W_{c_i}}] + m_{\mathrm{W}_i} d_i \hat{\boldsymbol{g}}_{\mathrm{s}_i}^T [\tilde{\boldsymbol{w}}_{2_i}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T \right] \dot{\boldsymbol{\omega}} \\
&\qquad + [J_{12_i} \mathrm{s}\theta_i + J_{13_i} \mathrm{c}\theta_i - m_{\mathrm{W}_i} d_i \ell_i \mathrm{s}\theta_i] \ddot{\gamma}_i + \left[J_{11_i} + m_{\mathrm{W}_i} d_i^2 \right] \dot{\Omega}_i \\
&= -\hat{\boldsymbol{g}}_{\mathrm{s}_i}^T \left[[I_{\mathrm{W}_i, W_{c_i}}]' \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i, W_{c_i}}] \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathrm{W}_i} d_i [\tilde{\boldsymbol{w}}_{2_i}] \left[2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \boldsymbol{r}_{W_{c_i}/B} \right] \right] \\
&\qquad + (J_{13_i} \mathrm{s}\theta_i - J_{12_i} \mathrm{c}\theta_i) \Omega \dot{\gamma} - m_{\mathrm{W}_i} d_i^2 \dot{\gamma}_i^2 \mathrm{c}\theta_i \mathrm{s}\theta_i + u_{\mathrm{s}_i}
\end{aligned} \tag{101}$$

Removing imbalance terms gives (recall that for the simplified case $\theta_i = 0$),

$$I_{\mathrm{W}_{s_i}} (\hat{\boldsymbol{g}}_{\mathrm{s}_i}^T \dot{\boldsymbol{\omega}} + \dot{\Omega}_i) = -I_{\mathrm{W}_{s_i}} \omega_t \dot{\gamma}_i + u_{\mathrm{s}_i} \tag{102}$$

NUMERICAL SIMULATION RESULTS

Numeric simulations are provided to demonstrate the fully-coupled imbalanced VSCMG model. Angular momentum is calculated to confirm that when no external disturbances are present angular momentum is conserved. System energy is calculated to show that when no external disturbances or internal torques are present, energy is conserved. With internal torques applied, the numerical energy rate of the fully-coupled and simplified models are compared to the theoretical value given by[16]

$$\dot{T}_{\mathrm{sc}} = \dot{\boldsymbol{r}}_{B/N}^T \boldsymbol{F} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T \boldsymbol{L}_B + \sum_{i=1}^{N} \dot{\gamma}_i u_{\mathrm{g}_i} + \sum_{i=1}^{N} \Omega_i u_{\mathrm{s}_i} \tag{103}$$

The simulations results directly compare the fully-coupled model to the simplified model using the formulation described in Reference 11. Simulation parameters used are given in Table 1. The scenario used to demonstrate the fully-coupled imbalanced VSCMG EOMs involves a rigid spacecraft hub and $N = 4$ VSCMGs. The lumped manufacturer imbalance parameters are related to the parameters used within this derivation using the imbalance parameter adaptation formulation given in Reference 17. This formulation allows us to assume the following:

$$d_i = \frac{U_{s_i}}{m_{\text{rw}_i}} \tag{104}$$

$$J_{13_i} = U_{d_i} \tag{105}$$

$$J_{12_i} = J_{23_i} = 0 \tag{106}$$

Figure 2 shows the attitude and body rates of the spacecraft for the duration of the simulation. In Figure 2(a), the spacecraft attitude computed using the simplified model is shown to rapidly drift from that of the fully-coupled model. The third MRP component in particular drifts in the opposite direction. This information is reflected in the spacecraft body rates as shown in Figure 2(b). The body rates as computed by the simplified model drift rapidly from those using the fully-coupled model, although the higher frequency variations are similar in amplitude. It is evident from this data that the body rates and attitude MRP of the spacecraft would likely be wildly different between the simplified and fully-coupled models if the simulation were propagated for longer than $t = 2$ seconds. Figure 3 shows the principle angle of the spacecraft with respect to inertial. Figure 3(a) shows the raw principle angle computed from the attitude MRP in Figure 2(a) and reflects much of the same information. After 2 second, the two models show principle angles that are different by several degrees. Figure 3(b) shows the higher frequency modes of the principle angle by subtracting out a polynomial fit of the data shown in Figure 3(a) to act as a high-pass filter of sorts.

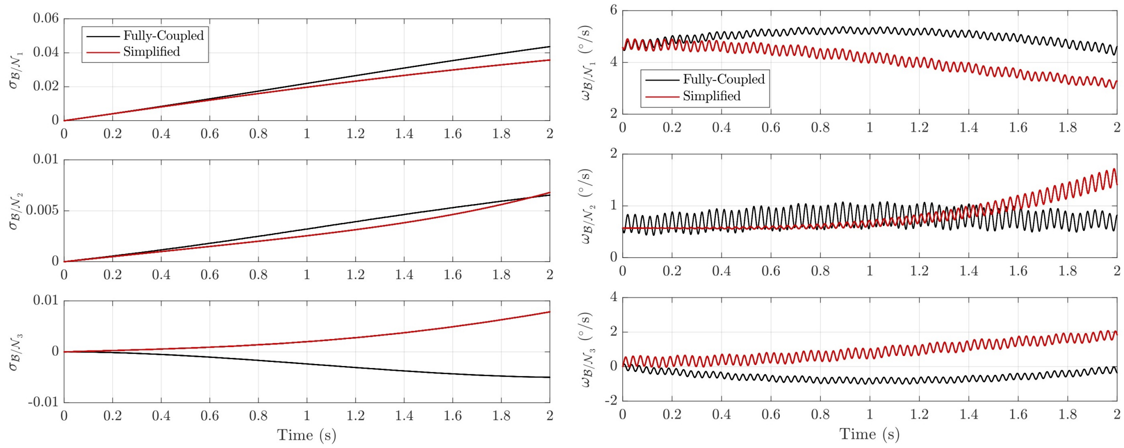
Figure 4 shows the translational position and velocity. These plots demonstrate that there is a non-zero effect due to VSCMG jitter on the position and velocity of the spacecraft. The position and velocity comparison of the fully-coupled and simplified model show that the simplified model is not able to track either position or angular velocity well for the given set of initial conditions. In Figure 4(b), it is evident that the simplified model has wildly underestimated the magnitude of the imbalance vibration effect on spacecraft velocity.

Figure 5 shows the VSCMG gimbal rate and wheel speeds. Again, it is evident that the simplified model has underestimated the effect of the vibration on each of the rates. The gimbal rate of VSCMG 1 in particular, shown in Figure 5(a), varies greatly between the two models. The fully-coupled model has a high frequency chatter with an amplitude of approximately 75 deg/s, whereas the simplified model shows no visible signs of chatter and slowly drifts in the same overall trend as the fully-coupled model. The wheel speed for gimbal 1, however, does not appear to closely match the same trend between the fully-coupled and simplified models. VSCMG 2 shows similar information regarding gimbal rate. The wheel speeds and gimbal rates agree with the time history of the applied wheel and gimbal torques as shown in Figures 6(b) and 6(a), respectively. The effect of the wheel torque is evident from looking at the wheel rates. The effect of the gimbal torque on the gimbal rates is not evident to the eye since the gimbal has a significantly larger moment of inertia.

Figure 7 shows the change in energy and angular momentum of the system for the fully-coupled and simplified models. Energy is plotted for a 1.5 second period since the wheel and gimbal torques are zero during this time and energy should be conserved. However, Figure 7(a) shows that using the simplified model causes energy to fluctuate whereas the fully-coupled model only includes integration error. Angular momentum, by definition, should be conserved for a closed system under the influence of internal torques, and is thus plotted for the entire duration of the simulation in Figure 7(b). It can be seen that the simplified model violates conservation of angular momentum and

Table 1. : Simulation parameters for the fully-coupled model. Note that wheel parameters apply to all wheels unless otherwise specified.

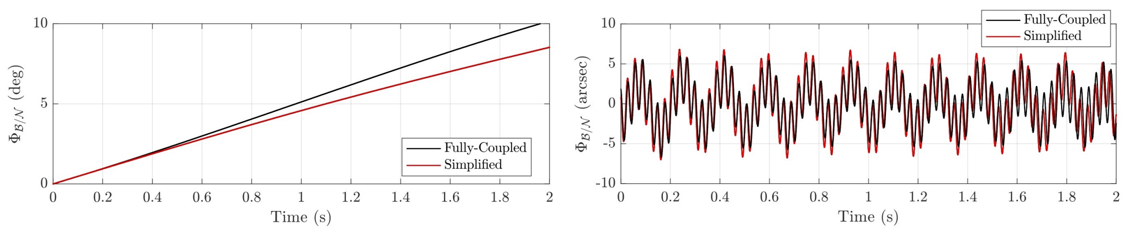
Parameter	Notation	Value	Units
Number of VSCMGs	N	4	-
Total spacecraft mass	m_{sc}	862	kg
Hub mass	m_{B}	750	kg
Wheel mass	m_{W}	4	kg
Gimbal mass	m_{G}	24	kg
Hub inertia tensor about hub CoM	$[I_{\text{hub},B_c}]$	${}^{\mathcal{B}}\begin{bmatrix} 900 & 4.15 & 2.93 \\ 4.15 & 800 & 2.75 \\ 2.93 & 2.75 & 600 \end{bmatrix}$	kg·m^2
Hub CoM location wrt B	$\boldsymbol{r}_{B_c/B}$	${}^{\mathcal{B}}\begin{bmatrix} -0.02 & 0.01 & 10 \end{bmatrix}^T$	cm
Wheel static imbalance	U_s	32	g·cm
Wheel dynamic imbalance	U_d	15.4	g·m^2
Wheel CoM offset (derived from U_s)	d	8.0	mm
Wheel inertia tensor about wheel CoM (derived from U_d)	$[I_{\text{W},W_c}]$	${}^{\mathcal{W}}\begin{bmatrix} 0.2 & 0 & 1.54\text{E}{-2} \\ 0 & 0.1 & 0 \\ 1.54\text{E}{-2} & 0 & 0.1 \end{bmatrix}$	kg·m^2
Gimbal inertia tensor about gimbal CoM (derived from U_d)	$[I_{\text{G},G_c}]$	${}^{\mathcal{W}}\begin{bmatrix} 9 & 0.81 & 0.24 \\ 0.81 & 11 & 0.93 \\ 0.24 & 0.93 & 5 \end{bmatrix}$	kg·m^2
VSCMG 1 location vector	$\boldsymbol{r}_{G_1/B}$	${}^{\mathcal{B}}\begin{bmatrix} -30 & 0 & 0 \end{bmatrix}^T$	cm
VSCMG 2 location vector	$\boldsymbol{r}_{G_2/B}$	${}^{\mathcal{B}}\begin{bmatrix} 30 & 0 & 0 \end{bmatrix}^T$	cm
VSCMG 3 location vector	$\boldsymbol{r}_{G_3/B}$	${}^{\mathcal{B}}\begin{bmatrix} 0 & -30 & 0 \end{bmatrix}^T$	cm
VSCMG 4 location vector	$\boldsymbol{r}_{G_3/B}$	${}^{\mathcal{B}}\begin{bmatrix} 0 & 30 & 0 \end{bmatrix}^T$	cm
Initial position	$\boldsymbol{r}_{B/N}$	${}^{\mathcal{N}}\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$	m
Initial velocity	$\boldsymbol{v}_{B/N}$	${}^{\mathcal{N}}\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$	m/s
Initial attitude MRP	$\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}}$	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$	-
Initial angular velocity	$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$	${}^{\mathcal{B}}\begin{bmatrix} 4.58 & 0.57 & 0 \end{bmatrix}^T$	deg/s
Initial wheel speeds	Ω	2000, 350, -11, 2	RPM
Initial wheel angles	θ	0, 0, 0, 0	deg
Initial gimbal speeds	$\dot{\gamma}$	-1.72, 0.63, 0, 0	deg/s
Initial gimbal angles	γ	0, 0, 0, 0	deg
Commanded wheel torques	u_s	0, 250, -250, 0	mN·m
Commanded gimbal torques	u_s	100, -100, 0, 0	mN·m



(a) Attitude MRP of the spacecraft for the fully-coupled and simplified models with $N = 4$

(b) Body rates of the spacecraft for the fully-coupled and simplified models with $N = 4$

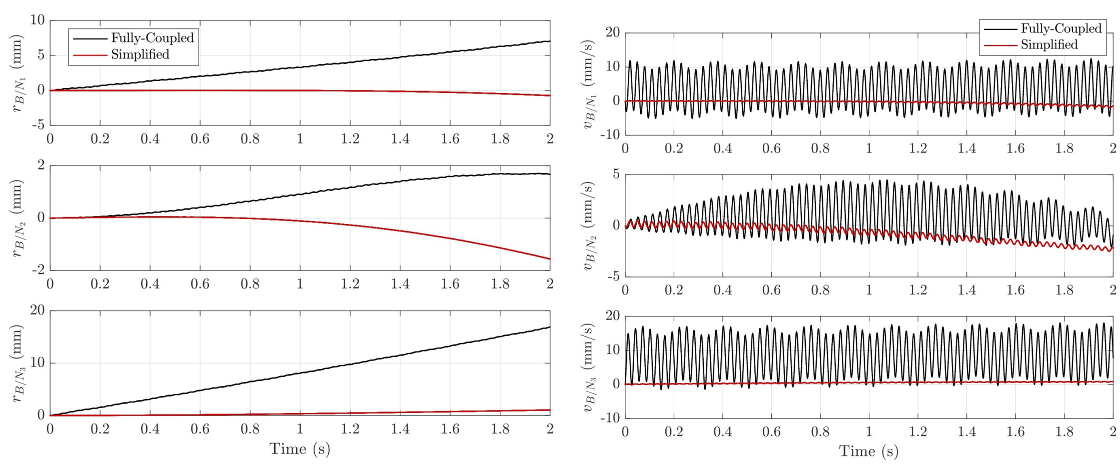
Figure 2. Attitude and body rates of the spacecraft



(a) Principal angle for the fully-coupled and simplified models with $N = 4$

(b) Principal angle jitter for the fully-coupled and simplified models with $N = 4$

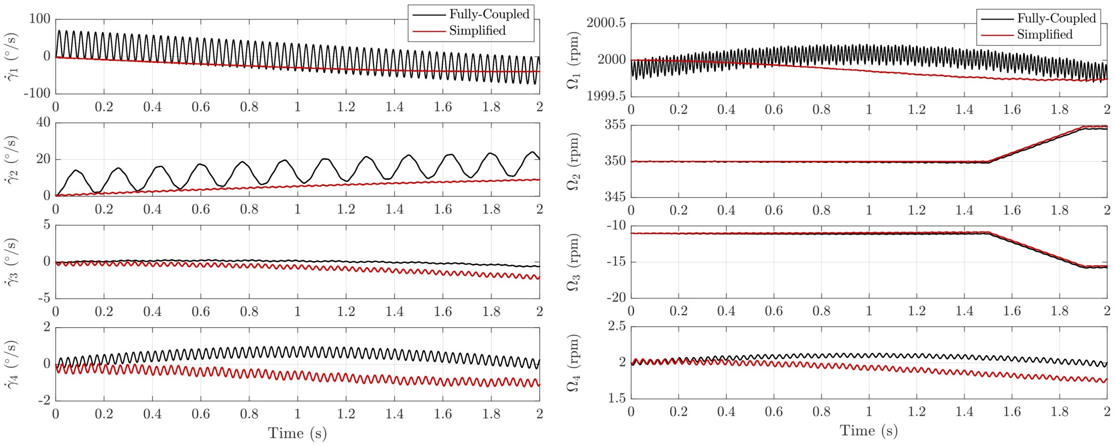
Figure 3. Principle angle and jitter of the spacecraft



(a) Inertial position of the spacecraft for the fully-coupled and simplified models with $N = 4$

(b) Inertial velocity of the spacecraft for the fully-coupled and simplified models with $N = 4$

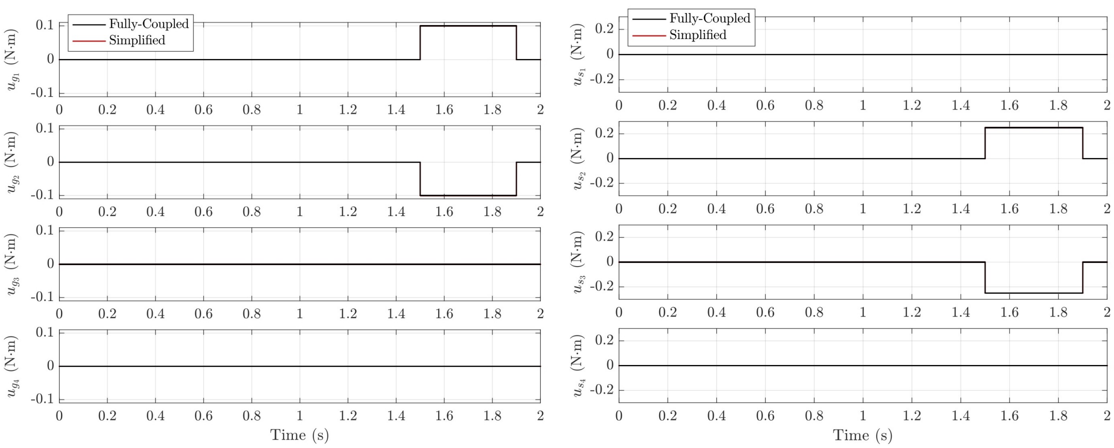
Figure 4. Position and velocity of the spacecraft



(a) Wheel speeds for the fully-coupled and simplified models with $N = 4$

(b) Open-loop wheel motor torques for the fully-coupled and simplified models with $N = 4$

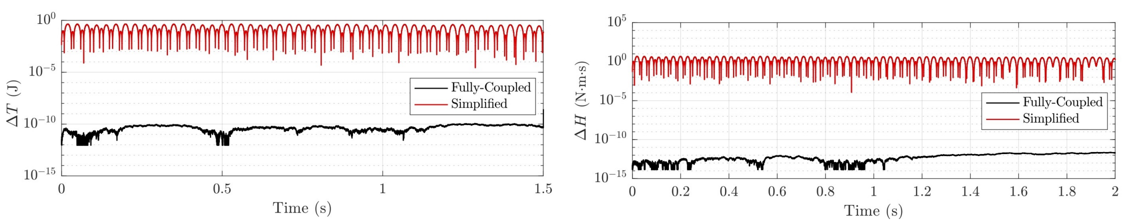
Figure 5. Wheel speed and gimbal rate of the VSCMGs



(a) Open-loop gimbal torques for the fully-coupled and simplified models with $N = 4$

(b) Open-loop wheel torques for the fully-coupled and simplified models with $N = 4$

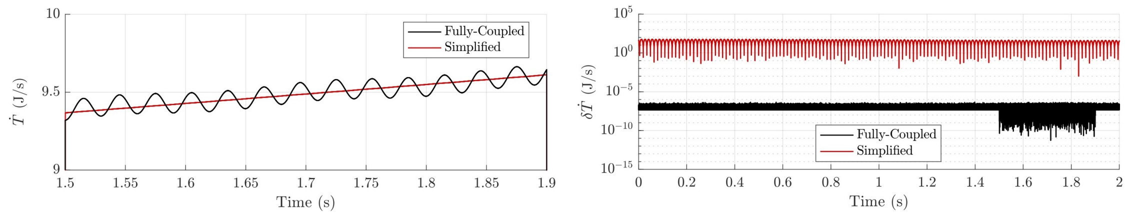
Figure 6. Wheel torque and gimbal torque of the VSCMGs



(a) System energy Δ for the fully-coupled and simplified models with $N = 4$

(b) System angular momentum Δ for the fully-coupled and simplified models with $N = 4$

Figure 7. Change in energy and angular momentum of the system



(a) System energy rate for the fully-coupled and simplified models with $N = 4$

(b) System energy rate error for the fully-coupled and simplified models with $N = 4$

Figure 8. Energy rate and energy rate error of the system

the fully-coupled model only exhibits integration error. For numerical simulations of a spacecraft, angular momentum and energy conservation is an important check to validate EOMs. For long simulation times the error in the simplified model will grow. This need for validation checks and error propagations are important characteristics to consider between both models.

Figure 8 shows the energy rate of and the energy rate error of the system for the fully-coupled and simplified models. Figure 8(a) shows the energy rate during the time period that the VSCMG has nonzero wheel and gimbal torques (from $t = 15$s to $t = 1.9$s), thus highlighting the difference between the fully-coupled and simplified models. The fully-coupled model has clearly visible fluctuation whereas the simplified model does not. Figure 8(b) shows the absolute error between the theoretical energy rate based on the internal torques and the numerically calculated energy rate based on numerically differentiating the energy T_{sc}. It is clear that the simplified model violates the theoretical energy rate, whereas the fully-coupled model has little error. Due to numerically differentiating the system energy T_{sc}, the comparison does show a larger error than in Figures 7(a)-7(b): approximately 10^{-7} compared to 10^{-12}.

CONCLUSIONS

Most previous work related to modeling jitter due to momentum exchange device (MED) imbalances models the effect as an external force and torque on the spacecraft. In reality, this effect is an internal force and torque on the spacecraft and thus requires a different formulation. The work presented in this paper develops the general fully-coupled model of variable-speed control moment gyroscope (VSCMG) imbalances. The fully-coupled model allows for momentum and energy validation to be implemented in a simulation.

Simulation results are provided to demonstrate the fully-coupled imbalance model compared to the simplified imbalance model. Energy is shown to be conserved when the motor torques are zero, and momentum is conserved throughout the length of the simulations. Energy rate is shown to closely match the theoretical energy rate. This provides validation of the fully-coupled models and highlights drawbacks to the simplified model, which violates conservation of momentum and energy. A comparison between the fully-coupled model and the simplified model shows that the imbalance parameter adaptation is adequate because the fully-coupled and simplified models give similar high-level results, for a fixed-axis scenario. However, because the simplified model is not valid in terms of conservation of energy and conservation of angular momentum it is undesirable when including additional complex dynamical models such as flexible dynamics or fuel slosh and causes error propagation to be a concern for lengthy simulation times. The research presented within this paper validated the EOMs using energy and angular momentum results from two completely independent software suites. Additionally, the states versus time were validated between the two simulations. This level of validation shows that the EOMs and software implementation method are correct beyond doubt. Implementations of the fully-coupled VSCMG model derived within this paper is to be released open-source in 2017 as part of the Basilisk astrodynamics software.*

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