LYAPUNOV OPTIMAL SATURATED CONTROL FOR NONLINEAR SYSTEMS

Rush D. Robinett and Gordon G. Parker
Sandia National Laboratories, Albuquerque, NM 87185

John L. Junkins and Hanspeter Schaub
Texas A&M University, College Station, TX 77843-3141

Simulated Reprint from

Journal of Guidance, Navigation and Control

Volume 20, Number 6, Nov.–Dec., 1997, Pages 1083-1088

A publication of the
American Institute of Aeronautics and Astronautics, Inc.
1801 Alexander Bell Drive, Suite 500
Reston, VA 22091

LYAPUNOV OPTIMAL SATURATED CONTROL FOR NONLINEAR SYSTEMS*

Rush D. Robinett† and Gordon G. Parker‡
Sandia National Laboratories, Albuquerque, NM 87185

Hanspeter Schaub§ and John L. Junkins¶
Texas A&M University, College Station, TX 77843

A generalized feedback control law design methodology is presented that applies to systems under control saturation constraints. Lyapunov stability theory is used to develop stable saturated control laws that can be augmented to any unsaturated control law that transitions continuously at a touch point on the saturation boundary. The time derivative of the Lyapunov function, an *error energy measure*, is used as the performance index which provides a measure that is invariant to the system dynamics. Constructive use of Lyapunov stability theory is used to establish stability characteristics of the closed-loop dynamics. Lyapunov optimal control laws are developed by minimizing the performance index over the set of admissible controls, which is equivalent to forcing the error energy rate to be as negative as possible.

I. Introduction

THE slewing of precision pointing spacecraft with reaction wheels has produced a need to address the stability of systems under saturated control. If a slew maneuver is performed that saturates one or more of the reaction wheels, does the spacecraft remain stable? Should the feedback gains be scaled back to keep all the controls in the unsaturated region or is it more beneficial to let some controls saturate and others operate unsaturated? These questions have led to the development of nonlinear optimal feedback control systems that are designed by Lyapunov's Direct Method and remain stable under saturated conditions.

The concept of optimal feedback controllers that are designed with Lyapunov stability theory can be found as far back in the literature as a homework problem in Reference 1, which originated in issues raised by Reference 2. Kalman and Bertram[2] introduced the idea of designing an optimal controller for a linear system that has saturation constraints. In this case, the controller design is performed for the entire state space region (i.e., both saturated and unsaturated). More present day applications include developments in References 3 and 4 where the concept of "Lyapunov Optimal" is utilized for feedback controller design. A control law is "Lyapunov Optimal" if it minimizes the first time derivative of the Lyapunov function over a space of admissible controls. More generally, a set of feedback gains are optimized by minimizing the tracking error of the feedback controller while tracking a specific reference maneuver.

In this paper, we employ the concept of Lyapunov optimality to design stable, saturated controllers for nonlinear systems. The Lyapunov function is the total error energy which for most mechanical systems is equivalent to an appropriate Hamiltonian function.[5] The performance index is the first time derivative of the Lyapunov function which is the instantaneous work rate and is a *kinematic relationship* independent of the system dynamics.[5] This truth is fundamental, because it means the structure of the controller depends only on the kinematic model and so the same control law stabilizes a large class of dynamical models. This means a high degree of robustness is *implicit* in this approach. A key region of interest is all of the state space where the controls are saturated. As a result of developments herein, the unsaturated control space is only considered when the saturated controller approaches the saturation boundary which may be a touch point.

*This work was sponsored by the United States Department of Energy under Contract DE-AC04-94AL85000.

†Distinguished Member of Technical Staff, Sandia National Laboratories, Senior Member AIAA.

‡Senior Member of Technical Staff, Sandia National Laboratories.

§Graduate Research Assistant, Aerospace Engineering Department, Student Member AIAA.

¶George Eppright Chair Professor, Aerospace Engineering Department, Fellow AIAA.

Upon passing through the saturation boundary, an unbounded optimal feedback controller can be employed in the unsaturated region. It is usually desirable, especially for flexible structures, that control transitions be continuous.

II. General Controller Design

Lyapunov's Direct Method is attractive to design globally asymptotically stable, nonlinear feedback controllers. The Lyapunov function is typically chosen as the total error energy of the system,

$$U = T + V \tag{1}$$

where T is the kinetic energy and V is the potential energy. The implicit reference state in Eq. (1) is the target state. U is typically positive definite; but when it is not, an additive fictitious energy function[3,4] equivalent to a position feedback loop with to-be-designed feedback gains can be used to modify U appropriately. Since most mechanical systems are natural systems, the Hamiltonian specializes for this case to the total system energy which motivates the alternative use of the Hamiltonian as a more general Lyapunov function candidate.[5] The reference state in Eq. (1) is the target state, thus U can be interpreted as the error energy of the system.

The first time derivative of the Lyapunov function in Eq. (1) is the instantaneous work rate

$$\dot{U} = \dot{H}(p,q) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} Q_i \tag{2}$$

where H is the Hamiltonian, $q = q(t)$ is the n-dimensional generalized coordinate vector, p is the n-dimensional generalized momentum vector, Q is the generalized force vector, and $\dot{()} \equiv d/dt()$. To be more specific,[6] L is defined as the system Lagrangian where

$$L = T(q,\dot{q}) - V(q) \tag{3}$$

with the classical definitions

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{4}$$

and

$$H(q,p) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q,\dot{q}) \tag{5}$$

leading to Hamilton's canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{6}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i \tag{7}$$

The combination of Eqs. (2)–(7) produces the power (work/energy) equation

$$\dot{U} = \sum_{i=1}^{n} \dot{q}_i Q_i \equiv \sum_{i=1}^{M} L_i \cdot \omega_i + \sum_{i=1}^{N} F_i \cdot \dot{R}_i \tag{8}$$

which is independent of the system dynamics. Eq. (8) is a *kinematic quantity* that applies to any system and is an ideal performance index for this problem.[7,8] In Eq. (8) $\{F_1, \ldots, F_N\}$ and $\{L_1, \ldots, L_M\}$ denote a set of forces and moments acting on a mechanical system. $\dot{R}_i$ denotes the inertial linear velocity of the point where F_i is applied. The component ω_i denotes the inertial angular velocity of the point where L_i is applied. It is important to note that Eq. (8) can be written immediately without further reference to the dynamical modeling assumptions and therefore holds for an infinity of systems. It is implicitly necessary, however, that the actual system must be controllable and the actual Hamiltonian must be positive-definite with respect to departures from the target state. Otherwise, it is necessary to establish sufficient insight to modify U and/or the number of control inputs.[3]

The goal of the controller design process is to choose a control law (i.e., select the equation form of the generalized forces) from an admissible set that will stabilize the system in an optimal fashion (i.e. make $\dot{U}$ as negative as possible). For saturated controls, the classical stabilizing controller takes the form

$$Q_i = -Q_{i_{max}} sgn(\dot{q}_i) \tag{9}$$

which is Lyapunov Optimal for the performance index

$$J = \dot{U} = \sum_{i=1}^{n} \dot{q}_i Q_i \tag{10}$$

The control law is optimal in a sense analogous to Pontryagin's Principle for optimal control because the controls are selected from an admissible set $|Q_i| \le Q_{i_{max}}$ such that the instantaneous work rate is minimized at every point in time. Two examples are solved in this section to demonstrate the generality of this feedback controller design process; subsequent examples of increasing dimensionality and generality are used to further illustrate the ideas.

Example 1: Duffing Oscillator

The first example is the design of a control law for a single degree of freedom nonlinear oscillator. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx + k_N x^3 = u \tag{11}$$

and the Lyapunov function (the system Hamiltonian of the unforced and undamped system) is

$$U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{4}k_N x^4 \tag{12}$$

The performance index is the time derivative of Eq. (12) and can be written immediately from Eq. (8) as

$$J = \dot{U} = \dot{x}Q = \dot{x}(-c\dot{x} + u) \tag{13}$$

For bounded control $|u| \leq u_{max}$, the performance index J is minimized by the feedback controller

$$u = -u_{max} sgn(\dot{x}) \tag{14}$$

Using this control law, $\dot{U}$ is reduced to the energy dissipation rate

$$\dot{U}(x, \dot{x}) = -c\dot{x}^2 - u_{max}\dot{x} \cdot sgn(\dot{x}) \tag{15}$$

It is of interest to note that an arbitrary, unknown, positive definite potential energy function $\Delta V(x)$ could be added to U in Eq. (12) and the negative gradient of ΔV to Eq. (11) — and *exactly the same result is obtained* for Eqs. (13) and (14). Thus the structure of the control law and the stability guarantee is invariant with respect to a large family of modeling assumptions.

Since $\dot{U}(x, \dot{x})$ of Eq. (15) is negative semi-definite, it can only be concluded at this point that the system is stable. Since admissible u, x and $\dot{x}$ will generally be bounded, Eq. (11) shows that $\ddot{x}$ will also be bounded. To prove asymptotic stability, the higher derivatives of U must be investigated.[9] The theory states that a sufficient condition to guarantee asymptotic stability is that the first non-zero higher order derivative of U, evaluated on the set of states such that $\dot{U}$ is zero, must be of *odd order* and negative definite. The only equilibrium point where $\dot{U}$ vanishes is $\dot{x} = 0$. The second derivative of U is

$$\frac{d^2U}{dt^2} = -2c\dot{x}\ddot{x} - u_{max}\ddot{x} sgn(\dot{x}) \tag{16}$$

which is zero for all x when $\dot{x} = 0$. The third derivative of U is

$$\frac{d^3U}{dt^3} = -2c\ddot{x}^2 - 2c\dot{x}\frac{d^3x}{dx^3} - u_{max}\frac{d^3x}{dx^3} sgn(\dot{x}) \tag{17}$$

Using Eq. (11) we find on the set where $\dot{x} = 0$ that

$$\left.\frac{d^3U}{dt^3}\right|_{\dot{x}=0} = -2\frac{c}{m}\left(kx + k_N x^3\right)^2 \tag{18}$$

which is a negative definite function of x. Therefore, the saturated control law in Eq. (14) is globally asymptotically stabilizing.

Example 2: Rigid Body Detumbling

The second example is detumbling a rigid body spacecraft to zero angular velocity at an unspecified orientation. The equations of motion are

$$I\dot{\omega} + \omega \times I\omega = u \tag{19}$$

where I is the matrix of principal moments of inertia, ω is the body- fixed angular rate vector instead of the generalized coordinate $\dot{q}$, and u is the control torque vector. Since the rigid body orientation is unspecified, we can consider ω to be the state vector, the system is of order three. The Lyapunov function is then the system kinetic energy

$$U = \frac{1}{2}\omega^T I\omega \tag{20}$$

and the time derivative of Eq. (20) is adopted as the performance index (energy rate):

$$J = \dot{U} = \omega^T u \tag{21}$$

We note Eq. (21) is simply the (kinematic) work/energy equation which we have written immediately. In this case, formal differentiation of Eq. (20), substitution for $I\dot{\omega}$ from Eq. (19), requires we recognize or verify that the work rate of the gyroscopic term is zero, i.e. that $\omega^T(\omega \times I\omega) = 0$. For more complicated dynamics, the use of Eq. (8) saves considerable algebra. Thus it is not necessary to reinvent the work/energy equation for each special case, we know it already. Obviously an infinite set of controls make $\dot{U}$ negative definite in Eq. (21), but for bounded controls $\{|u_i| < u_{max_i}\}$ the resulting Lyapunov optimal control law that minimizes Eq. (21) is clearly

$$u = -Asgn(\omega) \tag{22}$$

where we use the notational compaction

$$sgn(\omega) = (sgn(\omega_1), sgn(\omega_2), sgn(\omega_3))^T \tag{23}$$

and A is a positive definite diagonal weight matrix.

$$A = \begin{bmatrix} u_{max_1} & 0 & 0 \\ 0 & u_{max_2} & 0 \\ 0 & 0 & u_{max_3} \end{bmatrix} \tag{24}$$

This control law minimizes the performance index $\dot{U}$ to

$$J = \dot{U} = -\omega^T Asgn(\omega) \tag{25}$$

which is negative definite. Therefore u is a globally asymptotically stabilizing saturated control able to detumble a rigid body from any arbitrary rotation to rest.

At this point, it is important to note that mathematical difficulties[2] and practical system performance issues arise if these controllers are implemented directly for most systems. The discontinuity at the origin must typically be replaced with a region of unsaturated control to avoid chattering near $\omega = 0$. This unsaturated controller can either approximate the discontinuity or be some other stable/optimal feedback controller that transitions from the saturated controller on the saturation boundary. We restrict attention to control laws that transition continuously at the saturation boundary. The obvious choice is to approximate Eq. (9) with a linear controller of the type

$$Q_i = \begin{cases} -K_i \dot{q}_i & \text{for } |K_i \dot{q}_i| \leq Q_{i_{max}} \\ -Q_{i_{max}} sgn(\dot{q}_i) & \text{for } |K_i \dot{q}_i| > Q_{i_{max}} \end{cases} \tag{26}$$

or for this example with

$$u_i = \begin{cases} -k_i \omega_i & \text{for } |k_i \omega_i| \leq u_{max} \\ -u_{max} sgn(\omega_i) & \text{for } |k_i \omega_i| > u_{max} \end{cases} \tag{27}$$

where K_i and k_i are chosen feedback gains. This control continuously transitions across the saturation boundary and eliminates chattering. Note that Eqs. (26) and (27) allow some elements of the control vector to become saturated, while others are still in the unsaturated range. This differs from conventional gain scheduling and deadband methods which typically reduce the feedback gains to keep all controls in the unsaturated range.

III. Tracking Controller Design

To include the control problem for a slewing spacecraft, the design of tracking controllers that remain stable under saturated conditions must be considered. The formulation of the prior section is modified to accommodate tracking a reference motion $x_r(t)$ by rewriting Eq. (1) as the total tracking error energy

$$U_T = \Delta T + \Delta V \tag{28}$$

In this case, the concept of Lyapunov optimality is difficult to define since tracking stability cannot typically be guaranteed during the intervals while the controls are saturated. Nevertheless, globally asymptotically stable tracking controllers can often be achieved by generalizing the method used in the prior section. A generalized work/energy equation that is equivalent to Eq. (8) is not possible because the position and/or attitude error tracking coordinates are measured in a non-inertial reference frame. Also, consideration must be given to whether or not the prescribed trajectory is a feasible exact trajectory of the system. The following two examples, motivated by References 5 and 7, demonstrate the tracking feedback controller design process.

Example 3: Duffing Oscillator, Trajectory Tracking

The nonlinear oscillator of Eq. (11) is discussed first. Let the tracking error Δx be given as

$$\Delta x = x - x_r \tag{29}$$

The Lyapunov function that is interpreted as the total tracking error energy is defined as

$$U_T = \frac{1}{2} m \Delta \dot{x}^2 + \frac{1}{2} k \Delta x^2 + \frac{1}{4} k_N \Delta x^4 \tag{30}$$

The *reference trajectory* $x_r(t)$ is any smooth differentiable function. The first time derivative of Eq. (30) is

$$\dot{U}_T = \Delta \dot{x} \left[m \Delta \ddot{x} + k \Delta x + k_N \Delta x^3 \right] \tag{31}$$

which, making use of Eq. (11), and requiring Eq. (31) to be negative, leads to the following unsaturated feedback controller

$$\begin{aligned} u_{us} = & \left[m \ddot{x}_r + c \dot{x}_r + k x_r + k_N x_r^3 \right] \\ & + 3 k_N x x_r \Delta x - A \Delta \dot{x} \end{aligned} \tag{32}$$

and for saturated control, to minimize $\dot{U}_T$ in Eq. (31), we find

$$u = u_{max} sgn(u_{us}) \tag{33}$$

where A is a positive feedback gain. Using u_{us} from Eq. (32) in Eqs. (11) and (31), the performance index $\dot{U}_T$ becomes

$$\dot{U}_T = -(c + A) \Delta \dot{x}^2 \tag{34}$$

which is negative semi-definite. Therefore the tracking errors Δx and $\Delta \dot{x}$ will be stable and bounded in the absence of model errors. To investigate asymptotic stability using unsaturated control, let us investigate the higher derivatives of U_T. The only equilibrium point of $\dot{U}_T$ occurs where $\Delta \dot{x} = 0$. The second derivative of U_T is

$$\frac{d^2 U_T}{dt^2} = -2(c + A) \Delta \dot{x} \Delta \ddot{x} \tag{35}$$

which is zero at the equilibrium point, since $\Delta \ddot{x}$ is bounded. The third derivative of U_T is

$$\frac{d^3 U_T}{dt^3} = -2(c + A) \Delta \ddot{x}^2 - 2(c + A) \Delta \dot{x} \frac{d^3 \Delta x}{dt^3} \tag{36}$$

Using Eq. (11) the above can be evaluated on the set of states for which $\Delta \dot{x}$ vanishes as

$$\left. \frac{d^3 U_T}{dt^3} \right|_{\Delta \dot{x} = 0} = -2 \frac{(c + A)}{m} \left(k \Delta x + k_N \Delta x^3 \right)^2 \tag{37}$$

which is negative definite for any tracking error Δx. Thus the unsaturated feedback controller u_{us} is globally asymptotically stabilizing.[9]

It is prudent to evaluate the stability boundaries of the saturated control law of Eq. (33). The first step is to neglect the nonlinear terms and rewrite Eq. (33) as

$$u = u_{max} sgn([m\ddot{x}_r + c\dot{x}_r + kx_r] - A\Delta\dot{x}) \tag{38}$$

which imposes the sufficient stability constraint of

$$|m\ddot{x}_r + c\dot{x}_r + kx_r| \leq |u_{max}| \tag{39}$$

In other words, the required force to track the reference maneuver cannot exceed the maximum control authority. In hindsight this is an obvious result, if one expects to track an arbitrary reference trajectory. The second step is to include the nonlinear terms of Eq. (33) that produce the following sufficient stability constraint condition.

$$\begin{aligned} \big|\big(m\ddot{x}_r + c\dot{x}_r + kx_r + k_N x_r^3\big) \\ +3k_N x x_r \Delta x\big| \leq |u_{max}| \end{aligned} \tag{40}$$

Whereas the linear approximate result of Eq. (39) only depends on the absolute reference trajectory, Eq. (40) shows that the nonlinear stability boundary depends on the time history of the tracking error as well as the reference trajectory. One way to interpret the stability constraint of Eq. (40) is that the rate of growth of the tracking error under saturated conditions must be limited to the remaining control authority after accounting for the reference trajectory requirements. This may be illustrated by expressing the stability constraint of Eq. (40) as

$$|u_{us} + A\Delta\dot{x}| \leq |u_{max}| \tag{41}$$

Employing the triangle inequality yields the sufficient condition

$$|A\Delta\dot{x}| \leq |u_{max}| - |u_{us}| \tag{42}$$

which is a conservative stability constraint on $A\Delta\dot{x}$. Positive values of A cause the control to saturate earlier than if a control effort bound is simply placed on Eq. (32) for $A = 0$. However, under the saturated condition, the system is still stable so long as Eq. (40), or more conservatively Eq. (42), is satisfied. Since closed-loop stability is dependent on the reference maneuver and predicted dispersions off the reference maneuver, trajectory design is an iterative procedure.

Continuing with Example 3, system stability and trajectory design issues will be further exemplified by simulation results using numerical values for this example. The mass m is 1 kg, the stiffness k is 5 N/m, the nonlinear stiffness k_N is 25 N/m^3 and the damping c is 0.1 Ns/m. The velocity feedback gain A is 100 and the maximum controller effort u_{max} is set at 30 N. The reference trajectory adopted is

$$x_r = 0.5 + 0.4\sin(2\pi t \cdot 1.5Hz) \tag{43}$$

Notice that the maximum force required for the mass m to track x_r is larger than u_{max}, saturating the actual applied force. While the lack of compatibility between u_{max} and $x_r(t)$ is easily resolved by changing either, we consider this difficulty because we often face this situation in practice. To help illustrate the difference between control saturation and stability constraint violation, the time regions where the stability constraint of Eq. (40) is violated are greyed out in the following figures.

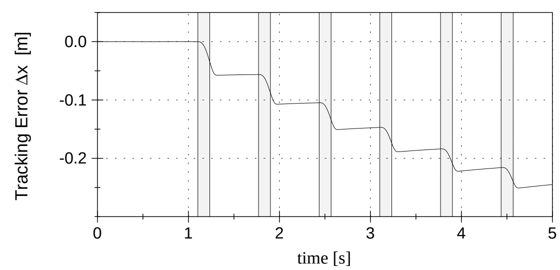

Fig. 1 Tracking Error

The tracking errors shown in Figure 1 grow during the intervals where the stability constraint is violated. Likewise, there is a slight decrease in tracking error between these unstable regions, consistent with the theoretical asymptotic stability for the unsaturated regions.

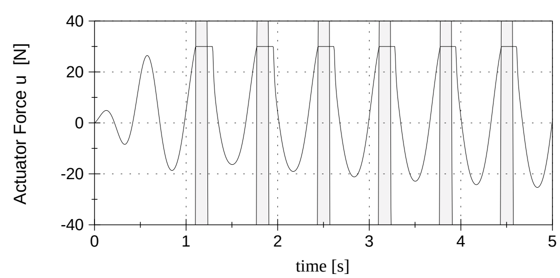

Fig. 2 Actuator Force

Figure 2 shows the control force u. Although the control clearly saturates, the stability constraint is violated only during subintervals of the torque saturation regions. A plot of $\dot{U}_T$ is shown in Figure 3. Recall that the stability constraint of Eq. (40) is derived from enforcing $\dot{U}_T$ to be negative. For this one-dimensional system the region where the stability inequality Eq. (40) fails corresponds directly with the region of positive $\dot{U}_T$. For higher dimensional systems, this type of stability condition will typically provide a much more conservative estimate of the stable region boundary than what the actual stability region boundary really is. This truth will be illustrated in the following example.

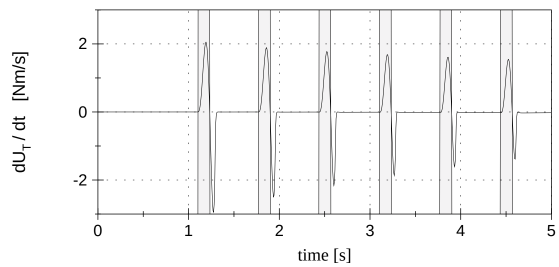

Fig. 3 Time Derivative of U_T

In practice, the reference maneuver can be designed with Eq. (40) in mind to allow an adequate margin for a finite Δx stability region. An important issue for practical applications is the dependence of the above process on knowledge of the system dynamical model.

Example 4: Detumbling Rigid Body to Specific State

Another tracking example deals with detumbling a rigid body and requiring it to track a prescribed reference trajectory. The body orientation relative to an inertial frame N is given through the 3x3 orientation (direction cosine) matrix $[BN]$. The reference orientation is given by the orientation matrix $[RN]$. The relative orientation between the actual and the reference orientation is given at any instant by $[BR]$=$[BN][RN]^T$. The attitude tracking error is then described by the modified Rodrigues parameter[10–13] vector σ, which minimally parameterizes the $[BR]$ matrix. Choosing σ to parameterize the attitude error is one of many possibilities, but as is evident below, this leads to a very attractive control law. Among many other advantages, these three parameters are non-singular for all possible $\pm 180^\circ$ rotations and have near-linear kinematics for up to $\pm 90^\circ$ rotations.[10,14,15] For tracking motions, we are virtually certain of always being in the near-linear range. This truth vastly expands the ranges of physical motions over which linear control theory (for gain design) is valid. The differential kinematic equation for σ is

$$\frac{d\sigma}{dt} = \frac{1}{2}\left[I\left(\frac{1-\sigma^T\sigma}{2}\right) + [\tilde{\sigma}] + \sigma\sigma^T\right]\delta\omega \tag{44}$$

Let ω be the actual body angular velocity written in the body fixed coordinate system and let ω_r be the reference body angular velocity written in the reference attitude coordinate system. Then the error in body angular velocity is

$$\delta\omega = \omega - [BR]\omega_r \tag{45}$$

The error in body angular acceleration is given by

$$\delta\dot{\omega} = \dot{\omega} - [BR]\dot{\omega}_r + \omega \times [BR]\omega_r \tag{46}$$

Let the Lyapunov function be defined as[10,11,14]

$$U_T = \frac{1}{2}\delta\omega^T I \delta\omega + 2K\log\left(1+\sigma^T\sigma\right) \tag{47}$$

where K is a scalar attitude feedback gain. Using the *log* function on a positive measure of tracking error in U_T results in the remarkable truth that σ appears linearly, without approximation, in $\dot{U}_T$.[11]

$$\dot{U}_T = \delta\omega^T I\delta\dot{\omega} + \delta\omega^T(K\sigma) \tag{48}$$

After using the expression for the body angular acceleration and substituting the equations of motion given earlier, $\dot{U}_T$ is reduced to

$$\begin{aligned}\dot{U}_T = \delta\omega^T &\left(-\omega\times(I\omega) + u - I[BR]\dot{\omega}_r\right.\\ &\left.+ I\omega\times([BR]\omega_r) + K\sigma\right)\end{aligned} \tag{49}$$

Note, since we are using a trajectory rather than a fixed point as the energy reference, we cannot write $\dot{U}_T$ immediately using Eq. (8). Let us define the unsaturated control torque u_{us} as

$$\begin{aligned}u_{us} = &I([BR]\dot{\omega}_r - \omega\times([BR]\omega_r))\\ &+\omega\times(I\omega) - K\sigma - P\delta\omega\end{aligned} \tag{50}$$

where P is a positive definite body angular velocity feedback gain matrix. This unsaturated control law reduces $\dot{U}_T$ of Eq. (49) to the non-positive quadratic form

$$\dot{U}_T = -\delta\omega^T P\delta\omega \tag{51}$$

and causes the closed-loop equations of motion to be the elegantly simple linear form

$$I\delta\dot{\omega} = -K\sigma - P\delta\omega \tag{52}$$

Since $\dot{U}_T$ is simply negative semi-definite, only stability and not asymptotic stability can be concluded. To prove that this unsaturated control law indeed leads to asymptotic stability, the higher order derivatives of U_T need to be investigated.[9] All points where $\dot{U}_T$ vanishes lie on the set Z where $\delta\omega = 0$. The theory states that the first non-zero higher order derivative of U_T must be of odd order and negative definite for the system to be asymptotically stable. The second derivative of U_T is

$$\frac{d^2U_T}{dt^2} = -\delta\dot{\omega}^T P\delta\omega - \delta\omega^T P\delta\dot{\omega} \quad (53)$$

which is zero on Z where $\delta\omega = 0$. The third derivative of U_T is

$$\frac{d^3U_T}{dt^3} = -\delta\ddot{\omega}^T P\delta\omega - 2\delta\dot{\omega}^T P\delta\dot{\omega} - \delta\omega^T P\delta\ddot{\omega} < 0 \quad (54)$$

After using Eq. (52), evaluating d^3U_T/dt^3 is

$$\left.\frac{d^3U_T}{dt^3}\right|_{\delta\omega=0} = -2\left(I^{-1}K\sigma\right)^T P\left(I^{-1}K\sigma\right) \quad (55)$$

which is negative definite on the set Z where $\delta\omega = 0$. Therefore u_{us} is a asymptotically stabilizing tracking control law.

Now assume that the available control torque about the i-th body axis is limited by u_{max_i}. Then following earlier analyses, we define a modified control law u as

$$u_i = \begin{cases} u_{us_i} & \text{for } |u_{us_i}| \leq u_{max_i} \\ u_{max_i} \cdot sgn(u_{us_i}) & \text{for } |u_{us_i}| > u_{max_i} \end{cases} \quad (56)$$

A conservative stability boundary (a sufficient condition for stability) for the above modified control torque is found to be

$$\begin{aligned} &|(I([BR]\dot{\omega}_r - \omega \times ([BR]\omega_r)) \\ &+\omega \times (I\omega) - K\sigma)_i| \leq |u_{max_i}| \end{aligned} \quad (57)$$

Note that for this higher dimensional system, this stability constraint may be overly conservative. The condition in Eq. (57) is violated if the inequality fails about any one axis. As will be shown in the following example, for higher dimensional cases the region where the stability inequality constraint is violated will typically be larger than the region of positive $\dot{U}_T$.

If zero reference motion is assumed from the beginning, then the above analysis leads to a globally asymptotically stable regulator for bounded torques. Thus if $\dot{x}_r(t) \to 0$ at some "final time," then thereafter the end game has global asymptotic stability. Following the above analysis, the unsaturated control torque becomes the simple linear control law

$$u_{us} = -K\sigma - P\delta\omega \quad (58)$$

and the modified control torque is the same as before in Eq. (56). Note that this control law guarantees to return a rigid body from any arbitrarily large errors in orientation and angular velocity to a specified orientation at rest, assuming of course that enough fuel is available. Thus global nonlinear controllability and stability are guaranteed. By using the modified control torque, reaching a u_{max_i} about one of the body axis does not affect whether or not $\dot{U}_T$ is negative, and thus does not affect the asymptotic stability. Due to the fact that the reference motion and model nonlinearity affects the structure of the unsaturated control law in Eq. (50) and the stability boundary in Eq. (57), the robustness of this approach to tracking controller design requires further study for each family of maneuvers and estimates of model uncertainty.

The control law in Eq. (56) is illustrated with the following numerical simulation. The diagonal inertia matrix has the entries are 385 kgm^2, 298 kgm^2 and 212 kgm^2. The reference maneuver is a smoothed near-minimum-time maneuver[14] starting at rest from the 3-1-3 Euler angles (-20°, 15°, 4°) to the angles (40°, 35°, 40°) with a final body angular velocity of ($0°/s$, $1°/s$, $0°/s$). This type of open-loop reference maneuver replaces any instantaneous torque switches with cubically splined ones. The final maneuver time is 112 seconds. The initial attitude error in 3-1-3 Euler angles is (1°, -2°, 1°). The initial body angular velocity error is ($-.025°/s$, $.1°/s$, $.025°/s$). The u_{max} vector containing the maximum available torque about each body axis is (0.15 Nm, 0.2 Nm, 0.15 Nm). The simulation was purposely chosen to periodically saturate the controls and violate the stability constraint of Eq. (57).

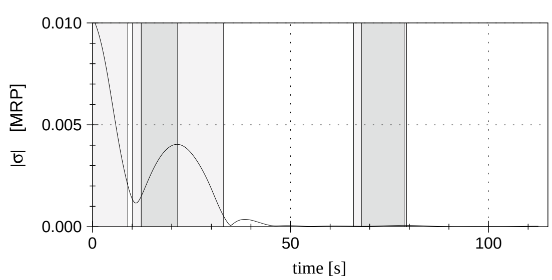

Fig. 4 Attitude Tracking Error Norm

As with the numerical simulation in Example 3, the time regions where Eq. (57) is violated due to torque saturation are shaded a light grey. As a comparison, the regions where $\dot{U}_T$ is actually positive are shaded a dark grey. As expected, the stability constraint violations of Eq. (57) are overly conservative and no longer coincide with areas of positive $\dot{U}_T$ as they did with the one-dimensional example previously. The maneuver is behaving in a stable fashion despite having large regions with stability constraint violations. Note that $\dot{U}_T$ is ultimately strictly negative after $t = 80$ seconds. Since Eq. (57) is very restrictive, the reference maneuvers for such torque saturated rotations often need to be designed interactively using Eq. (57) as an initial estimate.

The attitude tracking error σ is shown in Figure 4. The initial tracking error is reduced to almost zero by the maneuver end. This occurs despite the stability boundary being violated during two time spans. Keep in mind that having occasional excursions of $\dot{U}_T > 0$ does not guarantee instability, it simply cannot guarantee stability.

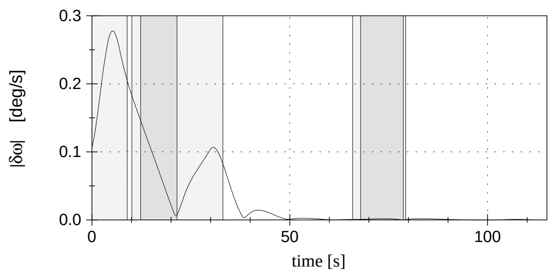

Fig. 5 Angular Velocity Tracking Error Norm

The angular velocity tracking error $\delta\omega$ is shown in Figure 5. It too is reduced to near-zero by maneuver end. The torque vector u is shown in Figure 6. As in Figure 2, the regions where $\dot{U}_T$ is positive is a subset of the regions where the torque about one of the body axis is saturated. Note also that not always are all three body axes saturated. The simulation contains cases where only one or two body axis are saturated. Note that the light grey stability constraint region of Eq. (57) extends over most of the torque saturated regions and even over some unsaturated regions. However, the dark grey regions of positive $\dot{U}_T$ only actually cover a smaller portion of the saturated torque cases. Covering some unsaturated regions further illustrates the conservative nature of the stability constraint in Eq. (57), since all unsaturated regions were shown to be asymptotically stable.

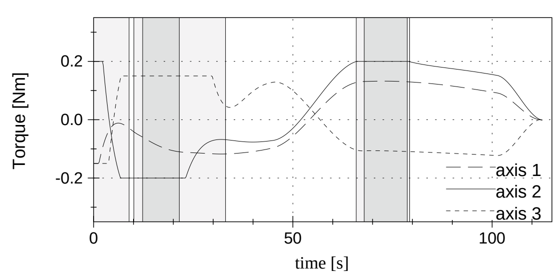

Fig. 6 Control Torque Vector

IV. Concluding Remarks

A study is presented of Lyapunov Optimality and the role of saturation constraints when using Lyapunov's Direct Method to design nonlinear feedback controllers for mechanical systems. For the special and usual case of controlled dynamics near a fixed point, how to efficiently design control laws and analyze global closed-loop stability properties is shown. These laws are robust with respect to dynamical model errors because they are derived from a kinematic work/energy principal. For the case of tracking-type controllers, the fixed point developments are extended and how to design controllers and analyze closed-loop stability using Lyapunov's Direct Method is illustrated. Certain difficulties and pitfalls are noted due to saturation constraints, conservativeness of stability sufficient condition, and robustness issues.

References

1. Bryson, A. E. and Ho, Y., *Applied Optimal Control.* Hemisphere Publishing Corporation, 1975.
2. Kalman, R. E. and Bertram, J. E., "Control System Analysis and Design Via the "Second Method" of Lyapunov: Continous Time Systems," *Transactions of ASME: Journal of Basic Engineering*, June 1960.
3. Junkins, J. L. and Bang, H., "Lyapunov Optimal Control Law for Flexible Space Structure Maneuver and Vibration Control," *Advances in Astronautical Sciences*, July–Aug. 1991. Paper AAS 91-143.
4. Kim, Y., Suk, T., and Junkins, J. L. "Optimal Slewing and Vibration Control of Smart Structures," *Smart Structures, Nonlinear Dyanamics and Control*, Vol. II. Prentice Hall, to appear in 1996.
5. Junkins, J. L. and Bang, H., "Manuever and Vibration Control of Hybrid Coordinate Systems Using Lyapunov Stability Theory," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 4, Jul.–Aug. 1993, pp. 668–676.
6. Meirovitch, L., *Methods of Analytical Dynamics.* McGraw–Hill, New York, 1970.
7. Oh, H. S., Vadali, S. R., and Junkins, J. L., "On the use of the work-energy rate principle for designing feedback control laws," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 15 No. 1, Jan–Feb 1992, pp. 272–277.
8. Juang, J., Wu, S., Phan, M., and Longman, R. W., "Passive Dynamic Controllers for Nonlinear Mechanical Systems," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 5, Sept.–Oct. 1993, pp. 845–851.
9. Junkins, J. L. and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures.* AIAA Education Series, Washington D.C., 1993.
10. Schaub, H. and Junkins, J. L., "Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters," *AAS/AIAA Spaceflight Mechanics Meeting*, Albuquerque, New Mexico, Feb. 13–16 1995. paper AAS 95–137.
11. Tsiotras, P., "New Control Laws for the Attitude Stabilization of Rigid Bodies," *IFAC Symposium on Automatic Control in Aerospace*, Palo Alto, CA, Sept. 12–16 1994, pp. 316–321.
12. Marandi, S. R. and Modi, V. J., "A Preferred Coordinate System and the Associated Orientation Representation in Attitude Dynamics," *Acta Astronautica*, Vol. 15, No. 15, 1987, pp. 833–843.

13. Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993. pp. 439–517.
14. Schaub, H., Robinett, R. D., and Junkins, J. L., "Globally Stable Feedback Laws for Near-Minimum-Fuel and Near-Minimum-Time Pointing Maneuvers for a Landmark-Tracking Spacecraft," *AAS/AIAA Astrodynamics Specialist Conference*, Halifax, Nova Scotia, Canada, Aug. 14–17 1995. paper AAS 95–417.
15. Schaub, H., Tsiotras, P., and Junkins, J. L., "Principal Rotation Representations of Proper NxN Orthogonal Matrices," *International Journal of Engineering Science*, Vol. 33, No. 15, Elsevier Science Ltd., Great Britain, 1995, pp. 2277–2295.