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Simplified Singularity Avoidance Using Variable Speed Control Moment Gyroscope Nullmotion

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I. Introduction

Singularity avoidance in Variable Speed Control Moment Gyroscope (VSCMG) systems can require significant computation to determine nullmotion steering commands. This paper presents a less complicated method of determining appropriate nullmotion steering laws while achieving similar performance as current, more complicated, methods.

Control Moment Gyroscope (CMG) clusters, which are often used for spacecraft attitude control, can encounter singular gimbal angle configurations where a general three-dimensional torque cannot be produced. Such singularities can be overcome through a variety of methods such as those presented in References 1 - 2. Another option to help avoid singularities is to use VSCMG devices, which allow a CMG device to vary its wheel speed, and thus produce a torque about two orthogonal axes (wheel spin and transfer axis).[3,4,5] A VSCMG cluster will not encounter gimbal locks (singular gimbal angle configurations) due to the reaction wheel modes. If all the CMG torque axes lie in a plane, then one of the reaction wheel control axis will point apart from this CMG-torque plane. Thus a VSCMG cluster can always produce the required torque of a chosen attitude control law without encountering temporary small attitude errors as the CMG singularity is avoided. However, using reaction wheel (RW) modes to drive through the CMG singular configuration requires significant RW motor torques, which is not power effective.[6] However, using the null space of the VSCMG system wisely can enable the system to avoid this singular CMG situation while completing the required control maneuvers. The extra RW control modes of the VSCMG allows for greater effectiveness to rearrange the gimbal angles away from the CMG singularity.[7] This increased null space of the VSCMG devices can also be used to create novel combined attitude and energy storage devices.[8,9,10] Here the rotor speed can be spun up during sun-lit portions of the orbit to store energy without changing the spacecraft attitude. Then, during a shaded orbit region the rotors can be spun down using the VSCMG null space to extract this energy again.

VSCMG steering laws lead to a simple condition which maps the desired rotor accelerations and gimbal rates into the required attitude control torque. The null space of this mapping is exploited by Schaub and Junkins using a gradient based method to drive the gimbal angles away from a CMG singularity.[7] The condition number of the mapping matrix is used as the singularity index. Yoon and Tsiotras analyze the CMG singularities of VSCMG devices in Reference 11 and provide a small modification to the gradient based null space proposed in Reference 7. The advantage of this modification is that stability of the singularity avoidance can be analytically guaranteed. However, this new null space steering law requires particular control of both the wheel speed and the gimbal rate, while the earlier method only required gimbal angle motion. The reduced actuation requirements are a benefit because this makes it easier for the null space to be used to implement auxiliary objects such as power storage demands, or returning the wheel speeds to their original values and avoiding long term rotor speed drift. Lee, Lee and Bang present in Reference 12 a general formulation to develop optimal null space VSCMG steering laws to avoid a CMG singularity. Their method can account for higher order CMG cost function sensitivities and provides analytical stability guarantees. However, as with the method by Yoon and Tsiotras, the VSCMG steering law dictates both rotor speed and gimbal angle changes. If reduced to a simple first-order form, their general formulation can be shown to be a generalization of the earlier methods discussed in References 7 and 11.

Many VSCMG null space steering methods have developed their formulation around the attitude regulation problem. The algebraic null space formulation often becomes significantly more complex if an attitude tracking problem is considered. This technical note investigates a simplified CMG singularity measure whose performance is equivalent to the previously published methods, but is implemented using a substantial reduction in complexity. In particular, considering an attitude tracking problems does not lead to an increase in complexity. The developments are performed, and numerically simulated, using the optimal steering formulation by Lee, Lee and Bang. However, the presented results could also be easily applied to the VSCMG null space steering law presented by Yoon and Tsiotras.

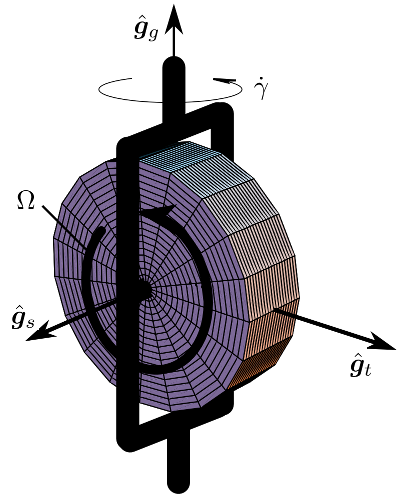

Figure 1: Variable-Speed CMG Coordinate Frame Illustration

II. VSCMG Steering Law Overview

Figure 1 illustrates the gimbal frame coordinate system $\mathcal{G} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{g}}_t, \hat{\boldsymbol{g}}_g\}$ used to describe the time varying orientation of the VSCMG relative to the spacecraft body $\mathcal{B}$. The gimbal rate $\dot{\gamma}_i$ is applied about the body-fixed axis $\hat{\boldsymbol{g}}_{g_i}$, while the rotor speed motor causes angular accelerations $\dot{\Omega}_i$ about the spin axis $\hat{\boldsymbol{g}}_{s_i}$. All VSCMG steering laws for both attitude regulation and tracking application leads to a control condition of the form:[3,4,5,13]

$$[Q]\dot{\boldsymbol{\eta}} = \mathbf{L}_r \tag{1}$$

$$[Q] = [D_0 \; D] \tag{2}$$

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} \dot{\boldsymbol{\Omega}} \\ \dot{\boldsymbol{\gamma}} \end{bmatrix} \tag{3}$$

Here $[D_0]$ and $[D]$ are $3 \times N$ matrices, with$N$ being the number of VSCMGs in the system. The projection matrix $[Q]$ is therefore a $3 \times 2N$ matrix which maps the VSCMG control states to the required $N \times 1$ control vector $\mathbf{L}_r$. This technical note follows the notation setup in References 7 and 5 which also provide expressions for $\mathbf{L}_r$ for attitude regulation and tracking control formulations. Finally, the parameters $\dot{\boldsymbol{\Omega}}$ and $\dot{\boldsymbol{\gamma}}$ are the $N \times 1$ wheel speed and gimbal rate vectors, respectively.

The wheel speed rate control matrix, $[D_0]$, is formulated by[5]

$$[D_0] = [\cdots \hat{\mathbf{g}}_{s_i} J_{s_i} \cdots] \tag{4}$$

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and the gimbal rate control matrix, $[D]$ for the reference attitude tracking problem is given by[5]

$$\begin{aligned}[D] = [\cdots J_{s_i}(\hat{\mathbf{g}}_{t_i}(\Omega_i + \frac{1}{2}\omega_{s_i}) + \frac{1}{2}\omega_{t_i}\hat{\mathbf{g}}_{s_i}) \\ - \frac{1}{2}J_{t_i}(\omega_{t_i}\hat{\mathbf{g}}_{s_i} + \omega_{s_i}\hat{\mathbf{g}}_{t_i}) + J_{g_i}(\omega_{t_i}\hat{\mathbf{g}}_{s_i} - \omega_{s_i}\hat{\mathbf{g}}_{t_i}) \\ + \frac{1}{2}(J_{s_i} - J_{t_i})(\hat{\mathbf{g}}_{s_i}\hat{\mathbf{g}}_{t_i}^T\boldsymbol{\omega}_r + \hat{\mathbf{g}}_{t_i}\hat{\mathbf{g}}_{s_i}^T\boldsymbol{\omega}_r)\cdots]\end{aligned} \tag{5}$$

where $\boldsymbol{\omega}_r$ is the reference trajectory angular velocity. The $\mathcal{G}$-frame axes can be grouped into a matrix such that each axis forms a column. For example, using the transverse axes $\hat{\boldsymbol{g}}_{t_i}$ we find

$$[G_t] = [\hat{\mathbf{g}}_{t_1} \cdots \hat{\mathbf{g}}_{t_i} \cdots \hat{\mathbf{g}}_{t_N}] \tag{6}$$

The same notation can be used to create $[G_s]$ and $[G_g]$ matrices. J_s, J_t and J_g are the combined wheel and gimbal structure moments of inertia in the spin, transverse and gimbal directions, respectively. The projection of the spacecraft body angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ onto the i^{th} gimbal frame $\mathcal{G}$ yields

$$\boldsymbol{\omega} = \omega_{s_i}\hat{\boldsymbol{g}}_{s_i} + \omega_{t_i}\hat{\boldsymbol{g}}_{t_i} + \omega_{g_i}\hat{\boldsymbol{g}}_{g_i} \tag{7}$$

If examining the attitude regulation problem, the VSCMG steering law projection matrix $[D]$ can be simplified from (5) by setting $\omega_r = 0$ and dropping non-working terms to become[5]

$$[D]_{reg} = [\cdots \hat{\mathbf{g}}_{t_i}(J_{s_i}(\Omega_i + \omega_{s_i}) - J_{t_i}\omega_{s_i})\cdots] \tag{8}$$

Note that both versions of $[D]$ depend the gimbal angles γ_i through the dependence on $\hat{\boldsymbol{g}}_{s_i}$, $\hat{\boldsymbol{g}}_{t_i}$, ω_{s_i} and ω_{t_i}, and on the rotor spin rates Ω_i. A further simplification of $[D]$ is typically made in recognizing that the $J_{t_i}\omega_{s_i}$ term is very small compared to the other terms due to the size of the inertias, and therefore the typical form for regulator control is,

$$[Q] = [D_0 \; D_1] \tag{9}$$

where

$$[D_1] = [\cdots \hat{\mathbf{g}}_{t_i} J_{s_i}(\Omega_i + \omega_{s_i})\cdots] \tag{10}$$

While the $[Q]$ matrix will never be singular, it is possible for the $[D]$ or $[D_1]$ matrices to become singular. In this case the VSCMG cannot fully employ the CMG mode and some RW modes must be employed to produce the required control torque $\mathbf{L}_r$. The goal of the VSCMG null motion steering law is to keep the $[D]$ or $[D_1]$ matrices (tracking or regulation cases) full rank at all times.

Solving the control constraint in Eq. (1) results in the spacecraft executing the desired stabilizing motion. But, due to the fact that a typical 4 VSCMG system has a 5-D null space, there are infinite ways to solve for the steering command $\dot{\boldsymbol{\eta}}$. Typically, the desired solution is executed primarily with the CMG modes using a weighted minimum norm inverse of Eq. (1). To ensure that the CMG mode can be utilized at all times happens, the control solution must consider CMG singularity avoidance using the VSCMG control modes.

III. Optimal Steering Law

Lee et al. present in Reference 12 an optimal VSCMG null space steering law formulation to avoid the CMG singular configuration. This formulation relies on a CMG singularity measure V which must be minimized while keeping the VSCMG steering commands $\dot{\boldsymbol{\eta}}$ small. This section provides a brief overview of Lee's method, including second order sensitivities. If the singularity measure V has a complex form, then the null space steering law formulation quickly increases in complexity. This technical note provides first and second order singularity sensitivities for regulation and tracking problems using both the $[D]$ matrix and simplified singularity measure.

The singularity avoidance cost function used to derive Lee's optimal steering is,[12]

$$J(\dot{\boldsymbol{\eta}}) = V(\boldsymbol{\eta} + \dot{\boldsymbol{\eta}}\Delta t) + \frac{1}{2}\dot{\boldsymbol{\eta}}^T W\dot{\boldsymbol{\eta}} + \frac{1}{2}(\dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}_d)^T Z(\dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}_d) \tag{11}$$

where $\dot{\boldsymbol{\eta}}_d$ is a vector of preferred gimbal and wheel speed rates.

$$\dot{\boldsymbol{\eta}}_d = \frac{\boldsymbol{\eta}_d - \boldsymbol{\eta}}{\Delta t} \tag{12}$$

where $\boldsymbol{\eta}_d$ is a vector of desired values for the gimbal angles and wheel speeds.

The first term V in the cost function J is the singularity avoidance cost. The second term is the weighted cost for the control effort, and the final term is the weighted cost for deviated state values. Typically only the wheel speeds are weighted in the third term, which allows the gimbals to vary in any way without impacting the cost function through this term. This is assumed to be the case for the remainder of this paper.

The Hessian matrix ($\bar{H}$) and the gradient vector ($\mathbf{g}$) are defined as

$$\bar{H} =\equiv \Delta t^2 V''(\boldsymbol{\eta}) = \Delta t^2 \begin{bmatrix} \frac{\partial^2 V}{\partial \Omega^2} & \frac{\partial^2 V}{\partial \Omega \partial \gamma} \\ \frac{\partial^2 V}{\partial \gamma \partial \Omega} & \frac{\partial^2 V}{\partial \gamma^2} \end{bmatrix} \tag{13}$$

and

$$\mathbf{g}(\boldsymbol{\eta}) = \begin{bmatrix} \mathbf{g}_\Omega \\ \mathbf{g}_\gamma \end{bmatrix} = \Delta t V'(\boldsymbol{\eta}) = \Delta t \begin{bmatrix} \frac{\partial V}{\partial \Omega} & \frac{\partial V}{\partial \gamma} \end{bmatrix}^T \tag{14}$$

where $\mathbf{g}_\Omega, \mathbf{g}_\gamma \in \mathbb{R}^N$ are the partitions of the gradient matrix. Using these definitions of the Hessian and the gradient matrices, and ignoring the higher order terms, the singularity avoidance cost is approximated as

$$V(\boldsymbol{\eta} + \dot{\boldsymbol{\eta}}\Delta t) \simeq V(\boldsymbol{\eta}) + \mathbf{g}^T\dot{\boldsymbol{\eta}} + \frac{1}{2}\dot{\boldsymbol{\eta}}^T\bar{H}\dot{\boldsymbol{\eta}} \tag{15}$$

Using this framework, the optimal steering law is solved to be,

$$\begin{bmatrix} \dot{\boldsymbol{\Omega}} \\ \dot{\boldsymbol{\gamma}} \end{bmatrix} = \hat{S}\hat{Q}^+\mathbf{L}_r + (\hat{S}\hat{Q}^+\hat{S}^T - \hat{H}^{-1}) \begin{bmatrix} \hat{\mathbf{g}}_\Omega \\ \hat{\mathbf{g}}_\gamma \end{bmatrix} \tag{16}$$

where

$$\hat{S} = \hat{H}^{-1}Q^T, \quad \hat{Q}^+ = (Q\hat{H}^{-1}Q^T)^{-1} \tag{17}$$

Here the modified Hessian and gradient matrices are defined as

$$\hat{H} = \bar{H} + W + Z \tag{18}$$

$$\hat{\mathbf{g}} = \begin{bmatrix} \hat{\mathbf{g}}_\Omega \\ \hat{\mathbf{g}}_\gamma \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Omega - Z_\Omega\dot{\Omega}_d \\ \mathbf{g}_\gamma \end{bmatrix} \tag{19}$$

It should be noted that this optimal steering law is very similar to that proposed in Reference 7 for singularity avoidance and constant wheel speeds. The main difference is that the optimal steering law uses the second order derivatives for V and requires both $\dot{\Omega}_i$ and $\dot{\gamma}_i$ to perform the desired null motion, whereas Reference 7 only uses the first order derivatives and the gimbal rates $\dot{\gamma}_i$. The choice of the singularity index V results in varying algebraic complexities of the optimal steering law formulation. Further, it is beneficial to not require both specific $\dot{\gamma}_i$ and $\dot{\Omega}_i$ to avoid a CMG singularity. This makes it simpler to implement other objectives such as nominally constant rotor speeds, or power extraction requirements, using the VSCMG null motion.

IV. Singularity Avoidance

Although the VSCMG configuration cannot become singular like a CMG system ($[Q]$ is always full rank),[3] avoiding the geometrically singular CMG configuration $\hat{\boldsymbol{g}}_{t_i}$ in the same plane for the regulation case) is still desirable. In general, it is more efficient to steer with the CMG mode as much as possible.[6] Furthermore, the rotor speeds will saturate much sooner than the gimbals when producing a given torque, therefore smart VSCMG control systems attempt to keep the wheel speeds near the nominal values.

Mathematically, the singular CMG situation that needs to be avoided is when the $\hat{\boldsymbol{g}}_{t_i}$ axis become co-planar. The gimbal rates $\dot{\gamma}_i$ produce a torque about the $\hat{\boldsymbol{g}}_{t_i}$ axis. Thus, if these only span a two-dimensional space, then the CMG control modes cannot produce a general three-dimensional vector. This is seen directly for the regulation case in the $[D_1]$ matrix; the columns depend directly on the transverse axes.

The same argument can be used for attitude tracking problem with the full $[D]$ matrix. Although there are other terms present that are proportional to $\hat{\boldsymbol{g}}_{s_i}$, the dominant terms are still the gyroscopic term proportional to $J_{s_i}\Omega_i$. If the transverse axes become co-planar, then torques out of plane could still be produced. But, because the other terms are small in comparison to the $J_{s_i}\Omega_i$ term, the required motion will be an undesirable method of producing the torque.

In any case, a VSCMG steering law not only implements the required torque, but also has some null motion to avoid the singularity situation. For the regulation case, Reference 7 continuously minimizes the condition number of the $[D_1]$ matrix. The condition number of a matrix, κ, is defined as the ratio of the largest to smallest singular values of the chosen 3xN matrix,

$$\kappa = \frac{\sigma_1}{\sigma_3} \tag{20}$$

For the tracking case one would use the condition number of the $[D]$ matrix. Likewise, these condition numbers can be used as the CMG singularity V function in the optimal steering law developed by Lee.[12]

This technical note proposes to always use the condition number of the $[G_t]$ matrix in place of the $[D]$ or $[D_1]$ matrix condition numbers. This makes intuitive sense; the situation that is to be avoided is the transverse axes $\hat{\boldsymbol{g}}_{t_i}$ becoming co-planar, and thus the loss of full 3-D CMG control authority. If the condition number of $[G_t]$ is kept small, then the matrix is being kept at full rank, and therefore the transverse axes are spanning 3-D space instead of becoming co-planar. Furthermore, using $[G_t]$ works for the regulation or tracking case, unlike $[D]$ or $[D_1]$, and therefore does not require reworking the steering law for each specific case.

Note the following subtle issue with using the condition number of $[G_t]$ versus that of $[D]$ or $[D_1]$. The later matrices depend on the rotor speeds Ω_i. Thus, the VSCMG null motion can modify the rotor speeds to minimize the condition number κ. This ability is lost with the simplified singularity measure using the condition number of $[G_t]$ only. However, as is illustrated in the enclosed numerical simulations, as long as the rotor speeds are kept apart from zero, the resulting CMG avoidance performance is essentially equivalent to those of using the more complex condition number formulation of the $[D_1]$ and $[D]$ matrices.

A second order form of Lee's optimal steering law is examined which requires the first and second partial derivatives of the condition number with respect to both the gimbal angles and wheel speeds. The first partial derivatives are defined as

$$\frac{\partial\kappa}{\partial\Omega_i} = \frac{1}{\sigma_3}\frac{\partial\sigma_1}{\partial\Omega_i} - \frac{\sigma_1}{\sigma_3^2}\frac{\partial\sigma_3}{\partial\Omega_i} \tag{21}$$

$$\frac{\partial\kappa}{\partial\gamma_i} = \frac{1}{\sigma_3}\frac{\partial\sigma_1}{\partial\gamma_i} - \frac{\sigma_1}{\sigma_3^2}\frac{\partial\sigma_3}{\partial\gamma_i} \tag{22}$$

and where $i = 1, \ldots, N$. The partial derivatives of the singular values of a given matrix, $[A]$, are given by[14]

$$\frac{\partial\sigma_j}{\partial\Omega_i} = \mathbf{u}_j^T\frac{\partial[A]}{\partial\Omega_i}\mathbf{v}_j \tag{23}$$

$$\frac{\partial\sigma_j}{\partial\gamma_i} = \mathbf{u}_j^T\frac{\partial[A]}{\partial\gamma_i}\mathbf{v}_j \tag{24}$$

where the SVD of $[A]$ is

$$[A] = [U][\Sigma][V]^T$$

$\mathbf{u}_j$ is the j^{th} column of $[U]$, and $\mathbf{v}_j$ is the j^{th} column of $[V]$.

The second partial derivatives of κ can be obtained by applying the chain rule to Eqs. (21) and (22) to get,

$$\begin{aligned}\frac{\partial^2\kappa}{\partial\Omega_i\partial\Omega_j} &= \frac{1}{\sigma_3}\frac{\partial^2\sigma_1}{\partial\Omega_i\partial\Omega_j} - \frac{1}{\sigma_3^2}\frac{\partial\sigma_1}{\partial\Omega_i}\frac{\partial\sigma_3}{\partial\Omega_j}\\ &\quad - \frac{1}{\sigma_3^2}\left(\frac{\partial\sigma_3}{\partial\Omega_i}\frac{\partial\sigma_1}{\partial\Omega_j} + \sigma_1\frac{\partial^2\sigma_3}{\partial\Omega_i\partial\Omega_j}\right)\\ &\quad + \frac{2\sigma_1}{\sigma_3^3}\frac{\partial\sigma_3}{\partial\Omega_i}\frac{\partial\sigma_3}{\partial\Omega_j}\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial^2\kappa}{\partial\Omega_i\partial\gamma_j} = \frac{\partial^2\kappa}{\partial\gamma_i\partial\Omega_j} &= \frac{1}{\sigma_3}\frac{\partial^2\sigma_1}{\partial\Omega_i\partial\gamma_j} - \frac{1}{\sigma_3^2}\frac{\partial\sigma_1}{\partial\Omega_i}\frac{\partial\sigma_3}{\partial\gamma_j}\\ &\quad - \frac{1}{\sigma_3^2}\left(\frac{\partial\sigma_3}{\partial\Omega_i}\frac{\partial\sigma_1}{\partial\gamma_j} + \sigma_1\frac{\partial^2\sigma_3}{\partial\Omega_i\partial\gamma_j}\right)\\ &\quad + \frac{2\sigma_1}{\sigma_3^3}\frac{\partial\sigma_3}{\partial\Omega_i}\frac{\partial\sigma_3}{\partial\gamma_j}\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial^2\kappa}{\partial\gamma_i\partial\gamma_j} &= \frac{1}{\sigma_3}\frac{\partial^2\sigma_1}{\partial\gamma_i\partial\gamma_j} - \frac{1}{\sigma_3^2}\frac{\partial\sigma_1}{\partial\gamma_i}\frac{\partial\sigma_3}{\partial\gamma_j}\\ &\quad - \frac{1}{\sigma_3^2}\left(\frac{\partial\sigma_3}{\partial\gamma_i}\frac{\partial\sigma_1}{\partial\gamma_j} + \sigma_1\frac{\partial^2\sigma_3}{\partial\gamma_i\partial\gamma_j}\right)\\ &\quad + \frac{2\sigma_1}{\sigma_3^3}\frac{\partial\sigma_3}{\partial\gamma_i}\frac{\partial\sigma_3}{\partial\gamma_j}\end{aligned} \tag{27}$$

The second partials of the singular values can then be determined by differentiating Eqs. (23) and (24),

$$\frac{\partial^2\sigma_j}{\partial\Omega_i\partial\Omega_k} = \mathbf{u}_j^T\frac{\partial^2[A]}{\partial\Omega_i\partial\Omega_k}\mathbf{v}_j \tag{28}$$

$$\frac{\partial^2\sigma_j}{\partial\Omega_i\partial\gamma_k} = \frac{\partial^2\sigma_j}{\partial\gamma_i\partial\Omega_k} = \mathbf{u}_j^T\frac{\partial^2[A]}{\partial\Omega_i\partial\gamma_k}\mathbf{v}_j \tag{29}$$

$$\frac{\partial^2\sigma_j}{\partial\gamma_i\partial\gamma_k} = \mathbf{u}_j^T\frac{\partial^2[A]}{\partial\gamma_i\partial\gamma_k}\mathbf{v}_j \tag{30}$$

For a given matrix $[A]$ (which will be $[D]$, $[D_1]$, or $[G_t]$), the partial derivatives will have the form:

$$\frac{\partial[A]}{\partial\gamma_i} = [\mathbf{0}\cdots\mathbf{0}\;\boldsymbol{\chi}_i\;\mathbf{0}\cdots\mathbf{0}] \tag{31}$$

$$\frac{\partial[A]}{\partial\Omega_i} = [\mathbf{0}\cdots\mathbf{0}\;\boldsymbol{\psi}_i\;\mathbf{0}\cdots\mathbf{0}] \tag{32}$$

$$\frac{\partial^2[A]}{\partial\Omega_i\Omega_j} = 0 \quad \forall i,j \tag{33}$$

$$\frac{\partial^2[A]}{\partial\Omega_i\gamma_j} = \frac{\partial^2[A]}{\partial\gamma_i\Omega_j} = [\mathbf{0}\cdots\mathbf{0}\;\boldsymbol{\xi}_i\;\mathbf{0}\cdots\mathbf{0}] \tag{34}$$

$$\frac{\partial^2[A]}{\partial\gamma_i\gamma_j} = [\mathbf{0}\cdots\mathbf{0}\;\boldsymbol{\phi}_i\;\mathbf{0}\cdots\mathbf{0}] \tag{35}$$

The derivation for each of the three matrices is shown in the following subsections. It is made clear that determining the singularity avoidance controls is much simpler for $[G_t]$ than either $[D]$ or $[D_1]$.

A. $[D_1]$ Matrix Condition Number

For the attitude regulation control case, Reference 7 presents the partial derivative of $[D_1]$ with respect to γ_i as,

$$\boldsymbol{\chi}_i = -\hat{\mathbf{g}}_{s_i}J_{s_i}(\Omega_i + \omega_{s_i}) + \hat{\mathbf{g}}_{t_i}J_{s_i}\omega_{t_i} \tag{36}$$

Likewise, the partial derivative of $[D_1]$ with respect to Ω_i is,

$$\boldsymbol{\psi}_i = \hat{\mathbf{g}}_{t_i}J_{s_i} \tag{37}$$

The second partial derivative of $[D_1]$ with respect to γ_i and Ω_i is,

$$\boldsymbol{\xi}_i = \begin{cases} 0 & i \neq j \\ -\hat{\mathbf{g}}_{s_i}J_{s_i} & i = j \end{cases} \tag{38}$$

The second partial derivative of $[D_1]$ with respect to γ_i is,

$$\boldsymbol{\phi}_i = \begin{cases} 0 & i \neq j \\ -\hat{\mathbf{g}}_{t_i}J_{s_i}(\Omega_i + 2\omega_{s_i}) - \hat{\mathbf{g}}_{s_i}J_{s_i}\omega_{t_i} & i = j \end{cases} \tag{39}$$

B. $[D]$ Matrix Condition Number

For the reference attitude tracking control case, the first partial derivative of $[D]$ with respect to γ_i is,

$$\begin{aligned}\boldsymbol{\chi}_i = J_{s_i}(&-\Omega_i\hat{\mathbf{g}}_{s_i} - \omega_{s_i}\hat{\mathbf{g}}_{s_i} \\ &+ \omega_{t_i}\hat{\mathbf{g}}_{t_i}) + J_{t_i}(\omega_{s_i}\hat{\mathbf{g}}_{s_i} - \omega_{t_i}\hat{\mathbf{g}}_{t_i}) \\ &+ (J_{s_i} - J_{t_i})(\hat{\mathbf{g}}_{t_i}\hat{\mathbf{g}}_{t_i}^T\boldsymbol{\omega}_r - \hat{\mathbf{g}}_{s_i}\hat{\mathbf{g}}_{s_i}^T\boldsymbol{\omega}_r)\end{aligned} \tag{40}$$

The partial derivative of $[D]$ with respect to Ω_i is,

$$\boldsymbol{\psi}_i = \hat{\mathbf{g}}_{t_i} J_{s_i} \tag{41}$$

The second partial derivatives of $[D]$ with respect to γ_i and Ω_i is,

$$\boldsymbol{\xi}_i = \begin{cases} 0 & i \neq j \\ -\hat{\mathbf{g}}_{s_i} J_{s_i} & i = j \end{cases} \tag{42}$$

The second partial derivative of $[D]$ with respect to γ_i is,

$$\boldsymbol{\phi}_i = \begin{cases} 0 & i \neq j \\ \begin{aligned} 2J_{s_i}\big(&-\frac{1}{2}\Omega_i\hat{\mathbf{g}}_{t_i} - \omega_{s_i}\hat{\mathbf{g}}_{t_i} - \omega_{t_i}\hat{\mathbf{g}}_{s_i}\big) \\ &+ 2J_{t_i}(\omega_{t_i}\hat{\mathbf{g}}_{s_i} + \omega_{s_i}\hat{\mathbf{g}}_{t_i}) \\ &- 2(J_{s_i} - J_{t_i})(\hat{\mathbf{g}}_{s_i}\hat{\mathbf{g}}_{t_i}^T\boldsymbol{\omega}_r + \hat{\mathbf{g}}_{t_i}\hat{\mathbf{g}}_{s_i}^T\boldsymbol{\omega}_r)\end{aligned} & i = j \end{cases} \tag{43}$$

C. $[G_t]$ Matrix Condition Number

For both the attitude regulation and tracking cases, the first partial derivatives of $[G_t]$ have the very simple form,

$$\boldsymbol{\chi}_i = -\hat{\mathbf{g}}_{s_i} \tag{44}$$

$$\boldsymbol{\psi}_i = \mathbf{0} \tag{45}$$

Likewise, the second partial derivatives have the simple form,

$$\boldsymbol{\xi}_i = \mathbf{0} \tag{46}$$

and

$$\boldsymbol{\phi}_i = \begin{cases} 0 & i \neq j \\ -\hat{\mathbf{g}}_{t_i} & i = j \end{cases} \tag{47}$$

The functional form of using the condition number of $[G_t]$ is much simpler than those of the $[D_1]$ or $[D]$ matrices. Furthermore, it is important to note that the wheel speeds are not needed to determine the nullmotion steering commands for singularity avoidance, unlike with the other cost functions. The effectiveness of this simple CMG singularity measure is demonstrated in the following numerical simulations.

V. Simulation Results

A test simulation is used to illustrate the performance of the singularity avoidance using $[G_t]$ for a tracking case. For this simulation, the satellite properties are the same as were used by Schaub,[7] and are reproduced in Table 1. This simulation uses the full nonlinear acceleration-based equations of motion developed in Reference 5. A sub-servo gimbal acceleration controller (with feed-forward term) is used to implement the VSCMG gimbal rate $\dot{\gamma}_i$ commands from the optimal steering law.

In the following simulation the optimal steering law proposed by Lee et al.[12] is used, and the control parameters are shown in Table 2. An additional VSCMG state goal is implemented where any rotor speeds Ω_i should return gradually back to their original states. The attitude control goal is to track a reference rotation while continuously avoiding a CMG singularity. The reference trajectory is created assuming the satellite is commanded to constantly point at a fixed spot on the Earth's surface. This is executed by rotating the spacecraft about it's first body axis $\hat{\boldsymbol{b}}_1$ in order to always keep the body-z axis pointing at the desired location.

Figure 2 shows attitude error between the body quaternion and the reference trajectory quaternion, and Figure 3 shows the body axis angular velocities along with the reference trajectory commands. The difference between the actual and reference trajectories is very small. Also recall that using any method of singularity avoidance results in the same tracking performance since the singularity avoidance motion is in the null space, and therefore does not apply any net torques to the spacecraft. The differences shown in the three illustrated cases are due to the different gimbal rate commands resulting in slightly different gimbal acceleration based implementations.

Table 1: Spacecraft properties.

Parameter	Value
I_{s_1}	15,053 kg-m^2/s
I_{s_2}	6,510 kg-m^2/s
I_{s_3}	11,122 kg-m^2/s
N	4
θ	54.75°
J_s	0.70 kg-m^2
J_t	0.35 kg-m^2
J_g	0.35 kg-m^2

Table 2: Control parameters.

Parameter	Value
$\Omega_i(t_0)$	628 rad/s
$\gamma_i(t_0)$	[45 -45 45 -45] deg
Δt	0.01 s
Ω_d	628 rad/s
W_γ	$0.8 I_{N\times N}$
W_Ω	$0.008 I_{N\times N}$
Z_Ω	$0.008 I_{N\times N}$
K	2 kg-m^2/s
$[P]$	$30 I_{N\times N}$ kg-m^2/s

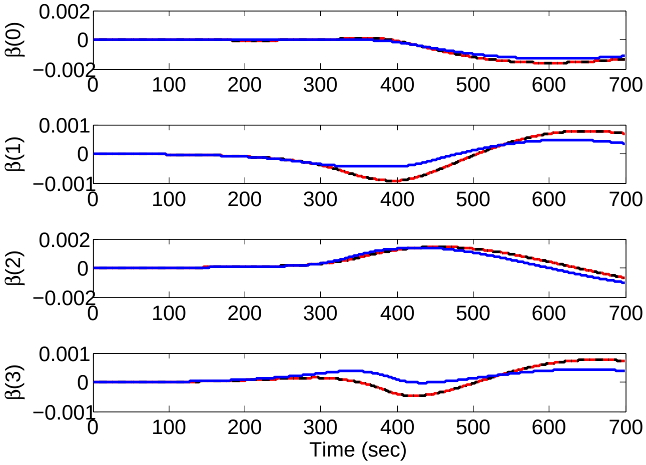

Figure 2: Quatornion attitude errors between body and reference frame. (- - -) illustrates the results using the $[G_t]$ cost function, (- - -) uses the $[D]$ cost function , and (—) illustrates the case without any CMG avoidance null motion.

Figures 4 and 5 show the VSCMG wheel speeds and gimbal angles, respectively, for the controller using the condition number of $[D]$ and $[G_t]$. Different singularity avoidance methods are expected to show different motions for each individual VSCMG component. However, this case shows that the results of using the different singularity avoidance measurements results in almost identical motion for every component of the system. This implies that the methods result in nearly identical null motion commands, and therefore the considerably more complex computations to use $[D]$ made little difference in the final closed-loop performance.

Finally, Figure 6 shows the comparison of the condition numbers for the three different singularity control methods. It is evident that the two optimal steering methods using either the condition number of $[D]$ or $[G_t]$ have nearly identical performance. The dif-

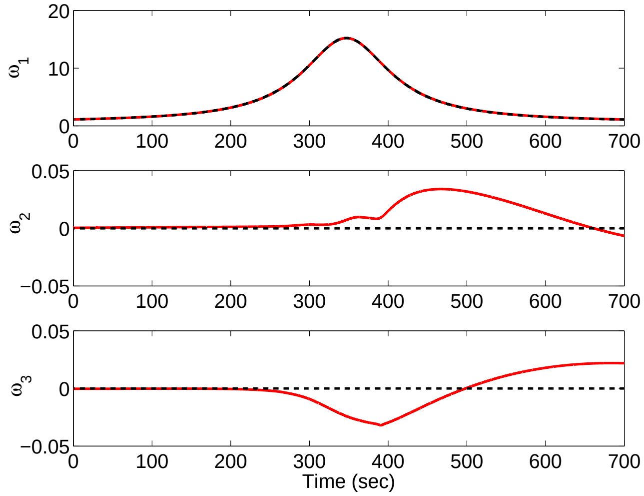

Figure 3: Spacecraft (—) and reference (- - -) angular velocities in mili-radians/second

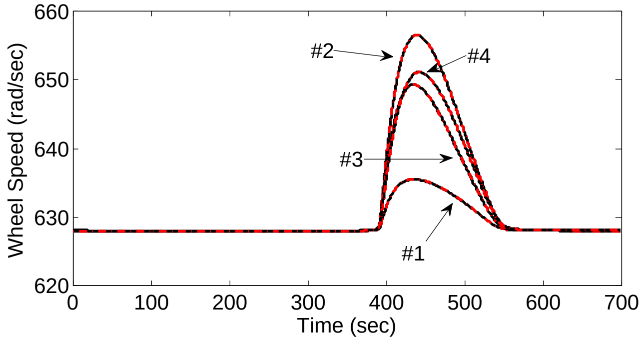

Figure 4: Time histories of the 4 VSCMG wheel speeds Ω_i. All four wheels show nearly identical speed profiles using $[G_t]$ (- - -) or $[D]$ (- - -)

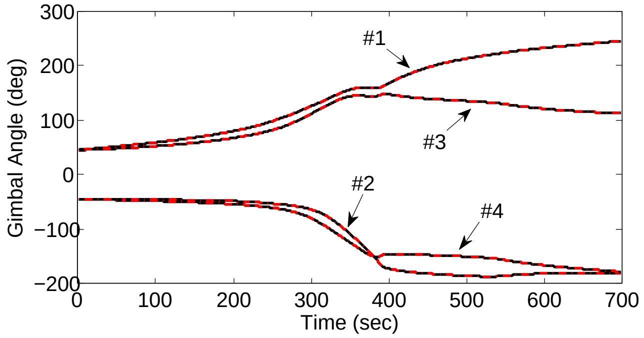

Figure 5: Time histories of the 4 individual VSCMG gimbal angles γ_i. All four wheels show nearly identical orientations using $[G_t]$ (- - -) or $[D]$ (- - -)

ferences is only 0.62% at the peak of κ at 450 seconds, too small to be of practical consequence. During the time frame when the condition number increases, the wheel speeds also change in Figure 4; this is when the VSCMG cluster is near the CMG singularity. Both singularity avoidance methods work appropriately to reduce the condition number, and then to return the rotor speeds back to their nominal values. The third simulation contrasts the optimal VSCMG steering law performance to the simple first order gradient method proposed in Reference 7. The condition number is also reduced back to a small value, but after growing first to a much larger value of approximately 82000. This comparison is interesting in that the optimal steering law minimizing the condition number of $[G_t]$ also only requires the gimbal rates in the resulting null space motion.

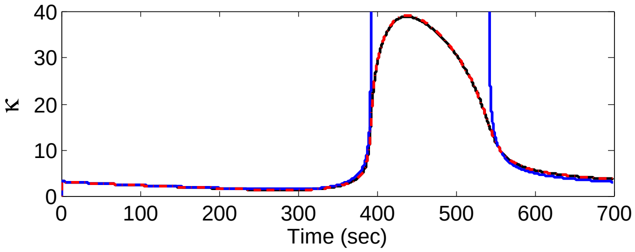

Figure 6: Condition numbers of using the optimal controller with $[D]$ (- - -) and $[G_t]$ (- - -). (—) illustrates the performance of the first order gradient method of Reference 7 using the condition number of $[D]$.

VI. Conclusion

This technical note introduces a simple method to implement a Control Moment Gyroscope (CMG) singularity avoidance null motion for a Variable Speed Control Moment Gyroscope (VSCMG) control system. This method is based on tracking the range of the transverse axes, instead of the rank of the VSCMG steering control projection matrix. The new approach does not require any knowledge of the rotor speeds in order to create singularity avoidance nullmotion steering commands. The performance using this simpler CMG singularity cost function is essentially identical to previously published methods, as is illustrated with a simple numerical simulation. The main benefit of this new method for CMG singularity avoidance using VSCMG devices is that the null motion commands are greatly simplified compared to previous methods.

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