Single-Axis Translating Effector

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1 Problem Statement

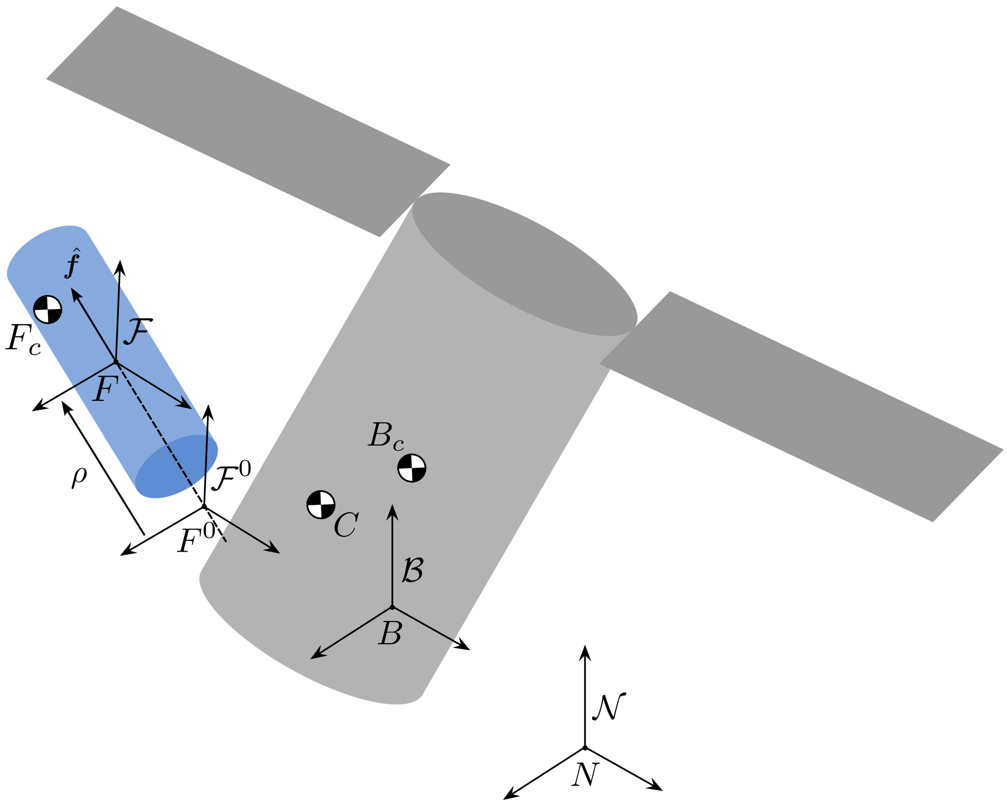

Fig. 1: Problem Statement.

The problem statement for the single-axis translating rigid body is illustrated in Figure 1. The inertial frame is represented by $\mathcal{N}$ with origin at point N. The spacecraft comprises a rigid body connected to a rigid hub through a single translation axis. The hub has a body-fixed frame $\mathcal{B}$ with origin B, and its center of mass is at point B_c. The mass of the hub is m_{hub}, and its inertia tensor about point B is $[I_{\mathsf{hub},B}]$. The translating rigid body has the $\mathcal{F}$ frame attached to it with its origin at point F. The center of mass of the effector is located at point F_c. The mass of the spinner is m_{F}, and its inertia tensor about its center of mass is $[I_{\mathsf{F},F_c}]$. The effector is parameterized by the variable ρ, which corresponds to the single-degree-of-freedom translation along the $\hat{\boldsymbol{f}}$ axis. The $\mathcal{F}^0$ frame, which has an origin point F^0, corresponds to the $\mathcal{F}$ frame when $\rho = 0$. The combined center of mass of the system is located at point C. The translation axis $\hat{\boldsymbol{f}}$ is constant, as seen by the $\mathcal{F}$ frame, and passes through points F and F^0.

2 Translational Equations of Motion

Using the Super Particle Theorem, we get

$$\boldsymbol{F}_{\mathsf{ext}} = m_{\mathsf{sc}}\ddot{\boldsymbol{r}}_{C/N} = m_{\mathsf{sc}}\ddot{\boldsymbol{r}}_{B/N} + m_{\mathsf{sc}}\ddot{\boldsymbol{c}} \tag{1}$$

where $\boldsymbol{c} \equiv \boldsymbol{r}_{C/B}$. Using the definition of the center of mass of the system, we get

$$m_{\mathsf{sc}}\boldsymbol{c} = m_{\mathsf{hub}}\boldsymbol{r}_{B_c/B} + m_{\mathsf{F}}\boldsymbol{r}_{F_c/B} \tag{2}$$

Using the transport theorem, we can express the inertial time derivative in body-frame derivatives as

$$\dot{\boldsymbol{c}} = \boldsymbol{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} \tag{3}$$

$$\begin{aligned}\ddot{\boldsymbol{c}} &= \boldsymbol{c}'' + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \dot{\boldsymbol{c}} \\ &= \boldsymbol{c}'' + 2\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times c)\end{aligned} \tag{4}$$

we can expand $\boldsymbol{r}_{F_c/B}$ into the following expression and then take derivatives

$$\boldsymbol{r}_{F_c/B} = \boldsymbol{r}_{F_c/F} + \boldsymbol{r}_{F/F^0} + \boldsymbol{r}_{F^0/B} = \boldsymbol{r}_{F_c/F} + \rho\hat{\boldsymbol{f}} + \boldsymbol{r}_{F^0/B} \tag{5}$$

$$\boldsymbol{r}'_{F_c/B} = \dot{\rho}\hat{\boldsymbol{f}} \tag{6}$$

$$\boldsymbol{r}''_{F_c/B} = \ddot{\rho}\hat{\boldsymbol{f}} \tag{7}$$

As for the body-frame time derivatives, we can take advantage of the fact that the $\boldsymbol{r}_{B_c/B}$ vector is fixed with respect to the $\mathcal{B}$ frame ($\boldsymbol{r}'_{B_c/B} = 0$) to get

$$m_{\mathsf{sc}}\boldsymbol{c}'' = m_{\mathsf{F}}\ddot{\rho}\hat{\boldsymbol{f}} \tag{8}$$

because $\hat{\boldsymbol{f}}$ is fixed in the $\mathcal{B}$ frame. Finally, we can combine all these terms to get

$$m_{\mathsf{sc}}\ddot{\boldsymbol{r}}_{B/\mathcal{N}} - m_{\mathsf{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{F}}\ddot{\rho}\hat{\boldsymbol{f}} = \boldsymbol{F}_{\mathsf{ext}} - 2m_{\mathsf{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\mathsf{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{9}$$

3 Rotational Equations of Motion

The rotational differential equation given about point B, which is not the system's center of mass, is given by

$$\dot{\boldsymbol{H}}_{\mathsf{sc},B} = \boldsymbol{L}_B + m_{\mathsf{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} \tag{10}$$

$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{F}/\mathcal{B}}$ because the translating body does not have any self-rotation. The angular momentum about point B is

$$\boldsymbol{H}_{\mathsf{sc},B} = \boldsymbol{H}_{\mathsf{hub},B} + \boldsymbol{H}_{\mathsf{fuel},B} \tag{11}$$

$$= [I_{\mathsf{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}]\dot{\boldsymbol{r}}_{B_c/B} + [I_{\mathsf{F},F_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\dot{\boldsymbol{r}}_{F_c/B} \tag{12}$$

The inertial time derivative of the total angular momentum can then be expressed as

$$\dot{\boldsymbol{H}}_{\mathsf{sc},B} = [I_{\mathsf{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathsf{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}]\ddot{\boldsymbol{r}}_{B_c/B} \tag{13}$$

$$+[I_{\mathsf{F},F_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathsf{F},F_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\ddot{\boldsymbol{r}}_{F_c/B} \tag{14}$$

The $m_{\mathsf{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}]\ddot{\boldsymbol{r}}_{B_c/B}$ term can be expressed as

$$m_{\mathsf{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}]\ddot{\boldsymbol{r}}_{B_c/B} = -m_{\mathsf{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{B_c/B} \tag{15}$$

Using the Jacobi identity to rewrite the above equation allows it to be combined into an inertia term in the total angular momentum equation. The same can be done for the $m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\ddot{\boldsymbol{r}}_{F_c/B}$ term to produce

$$\dot{\boldsymbol{H}}_{\mathsf{sc},B} = [I_{\mathsf{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\ddot{\rho}\hat{\boldsymbol{f}} + 2m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{F_c/B} \tag{16}$$

The final term can be split, and once again, using the Jacobi identity, can be written in terms of $[I'_{\mathsf{F},B}]$.

$$2m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{F_c/B} = -[I'_{\mathsf{F},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - m_{\mathsf{F}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{F_c/B}]\boldsymbol{r}'_{F_c/B} \tag{17}$$

Combining these results into the rotational equation of motion and equating it to the first relation for angular momentum, we get

$$\begin{aligned}m_{\mathsf{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\mathsf{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\ddot{\rho}\hat{\boldsymbol{f}} &= \boldsymbol{L}_B \\ -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} &- [I'_{\mathsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - m_{\mathsf{F}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{F_c/B}]\boldsymbol{r}'_{F_c/B}\end{aligned} \tag{18}$$

4 Effector Equation of motion

The equation of motion for the effector is

$$m_{\mathsf{F}}\ddot{\boldsymbol{r}}_{F_c/N} = \boldsymbol{F} \tag{19}$$

Beginning with $\ddot{\boldsymbol{r}}_{F_c/N}$, it can be expressed as

$$\boldsymbol{r}_{F_c/N} = \boldsymbol{r}_{B/N} + \boldsymbol{r}_{F_c/B} \tag{20}$$

$$\dot{\boldsymbol{r}}_{F_c/N} = \dot{\boldsymbol{r}}_{B/N} + \boldsymbol{r}'_{F_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{F_c/B} \tag{21}$$

$$\ddot{\boldsymbol{r}}_{F_c/N} = \ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{r}''_{F_c/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{F_c/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{F_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{F_c/B}) \tag{22}$$

Noting that $\boldsymbol{r}''_{F_c/B} = \ddot{\rho}\hat{\boldsymbol{f}}$, we get

$$m_{\mathsf{F}}\hat{\boldsymbol{f}}\ddot{\rho} = \boldsymbol{F} - m_{\mathsf{F}}\ddot{\boldsymbol{r}}_{B/N} + m_{\mathsf{F}}\boldsymbol{r}_{F_c/B} \times \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - 2m_{\mathsf{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{F_c/B} - m_{\mathsf{F}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{F_c/B}) \tag{23}$$

Since a minimal coordinate set is used, equation(19) is projected onto the minimal coordinate space, which means that both sides are dotted with $\hat{\boldsymbol{f}}$, which yields:

$$\ddot{\rho} = -\hat{\boldsymbol{f}}^T\ddot{\boldsymbol{r}}_{B/N} + \hat{\boldsymbol{f}}^T[\tilde{\boldsymbol{r}}_{F_c/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{f}}^T\frac{\boldsymbol{F}}{m_{\mathsf{F}}} - 2\hat{\boldsymbol{f}}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{F_c/B} - \hat{\boldsymbol{f}}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{F_c/B} \tag{24}$$

5 Backsubstitution

The Backsubstitution Method is defined in Reference 1. The effector contributions are

$$\boldsymbol{a}_\rho = -\hat{\boldsymbol{f}} \tag{25}$$

$$\boldsymbol{b}_\rho = [\tilde{\boldsymbol{r}}_{F_c/B}]\hat{\boldsymbol{f}} \tag{26}$$

$$c_\rho = -2\hat{\boldsymbol{f}}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{F_c/B} - \hat{\boldsymbol{f}}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{F_c/B} \tag{27}$$

The hub contributions are

$$[A] = m_{\text{SC}}[I_{3\times3}] - m_{\mathsf{F}}\hat{\boldsymbol{f}}\boldsymbol{a}_\rho^T \tag{28}$$

$$[B] = -m_{\mathsf{sc}}[\tilde{\mathbf{c}}] - m_{\mathsf{F}}\hat{\boldsymbol{f}}\boldsymbol{b}_\rho^T \tag{29}$$

$$[C] = m_{\mathsf{sc}}[\tilde{\mathbf{c}}] + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\hat{\boldsymbol{f}}\boldsymbol{a}_\rho^T \tag{30}$$

$$[D] = [I_{\mathsf{sc},B}] + m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\hat{\boldsymbol{f}}\boldsymbol{b}_\rho^T \tag{31}$$

and

$$\mathbf{v}_{\mathsf{trans}} = \boldsymbol{F}_{\mathsf{ext}} - 2m_{\mathsf{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c}' - m_{\mathsf{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\mathbf{c} - m_{\mathsf{F}}\hat{\boldsymbol{f}}c_\rho \tag{32}$$

$$\mathbf{v}_{\mathsf{rot}} = \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\mathsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\mathsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - m_{\mathsf{F}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{F_c/B}]\boldsymbol{r}'_{F_c/B} - m_{\mathsf{F}}[\tilde{\boldsymbol{r}}_{F_c/B}]\hat{\boldsymbol{f}}c_\rho \tag{33}$$

References

[1] Cody Allard, Manuel Diaz-Ramos, Patrick W. Kenneally, Hanspeter Schaub, and Scott Piggott. Modular software architecture for fully-coupled spacecraft simulations. *Journal of Aerospace Information Systems*, 15(12):670–683, 2018.