STABILITY AND RECONFIGURATION ANALYSIS OF A CIRCULARY SPINNING 2-CRAFT COULOMB TETHER

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IEEE Aerospace Conference

Big Sky, MT March 3–10, 2007

Stability and Reconfiguration Analysis of a Circularly Spinning 2-Craft Coulomb Tether

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Abstract—The concept of a spinning 2-craft Coulomb tether is introduced. Here a physical tether is replaced with an electrostatic force field resulting in an attractive Coulomb force between the 2 craft. The spacecraft charge is assumed to be regulated with an active charge servo system. The open-loop stability of a Coulomb tether with constant spacecraft charges is investigated. The reduced equations of motion for a deep space mission are obtained and linearized to determine eigenvalues of the perturbed motion. This analysis shows that if the plasma Debye length is smaller than the spacecraft separation distance the radial motion is guaranteed to be unstable. For larger Debye lengths the nonlinear radial motion is locally stable. The perturbed out-of-plane motion is shown to always be stable regardless of Debye length. Further, open-loop charge solutions are obtained to perform reconfiguration where the circular orbit radius is changed to a new value. This maneuver is related to the classical Hohmann transfer orbit between circular orbits. However, in the Coulomb tether concept the reconfiguration is achieved by varying the effective gravitational parameter through spacecraft charge changes.

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1. INTRODUCTION

Coulomb thrusting is a novel method to control the close relative motion of spacecraft using electrostatic (Coulomb) force fields. This concept was introduced by King and Parker in [1] in 2001 where they explored the natural spacecraft charging that occurred on a GEO satellite. Instead of treating this Coulomb force as a perturbation, they proposed to use it instead as an active means of relative motion control. Through active charge emission of electrons and ions the spacecraft potential is regulated to desired values.

This has led to a multitude of novel relative motion missions. The concept of virtual Coulomb structures has the electrostatic forces perfectly cancel out the differential gravitational acceleration, resulting in a spacecraft cluster whose satellite positions appear frozen as seen by the rotating chief local-vertical-local-horizontal (LVLH) frame [1], [2], [3], [4]. However, all charged static relative equilibria solutions in orbit or in deep space have been unstable and will required active charge feedback to stabilize.

The first feedback stabilized charged virtual structures is the nadir aligned Coulomb tether concept discussed in [5] and [6]. Here the physical tether connecting 2 spacecraft is replaced with a Coulomb force field. However, while a physical tether must always be in tension, the Coulomb tether can exert both attractive and repulsive forces between the 2 craft. But, while the cable tether can have lengths of multiple kilometers, the Coulomb tether concept is only applicable for relative small separation distances of up to 100 meters. The electrostatic force field strength drops off rapidly with increasing separation distances, limiting its effectiveness to relatively close mission scenarios of less than 100 meters. The potential Coulomb tether applications include deploying a free-flying sensor and tethering it to the mother craft using Coulomb forces, or using this electrostatic force to achieve a rendezvous and docking approach. Natarajan shows in [5] that sensing only the separation distance is sufficient to develop a charge feedback control law which asymptotically stabilizes both the separation distance and the in-plane motion. The out-of-plane motion for the nadir aligned Coulomb tether is naturally stabilized through the gravity gradient torque acting across the cluster.

The first spinning charged spacecraft clusters are explored in [7]. Here the charged 3-body problem is written in a form similar to Lagrange's invariant shape gravitational 3-body problem yielding the famous collinear and triangular libration points. Similar results are obtained for the spinning 3 charged spacecraft problem. However, this analysis only investigates the open-loop charges required for a relative equilibria and discusses the resulting trajectory shapes and boundedness. The stability of any spinning charged spacecraft cluster has not yet been explored.

Beyond looking at electrostatic forces fields to control satellite relative motion, MIT is investigating the use of electromagnetic force fields to control the satellite relative orbits [8]. This concept can produce general force vectors between the

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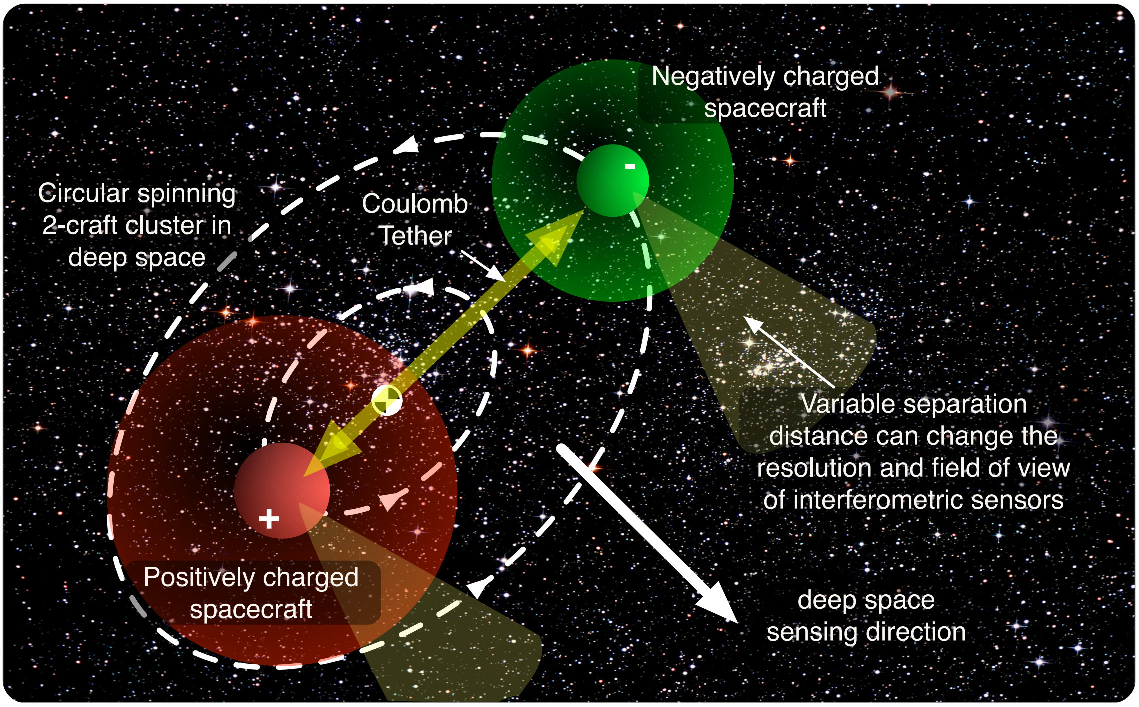


Figure 1. Illustration of a 2-craft Coulomb tether formation spinning with a constant separation distance in deep space.

craft, but requires sophisticated magnetic coils and an active attitude control system to absorb momentum. Mason Peck has also looked at using electrostatically charged spacecraft control [9]. However, he is looking to exploit the Lorentz force which arises from a charged body flying through a planets magnetic field.

This paper investigates the orbital stability of a spinning charged 2-craft cluster as illustrated in Figure 1. The spinning Coulomb tether formation contains spacecraft with opposite charges, resulting an attractive force which balances the centripetal force. This configuration is a constant charge relative equilibria solution of this system. Of interest is the question of whether this system is stable under small position, velocity or spacecraft charge errors. Another question of interest is: How can charge be used to vary the circular trajectory radii and reconfigure the shape of the spinning 2-craft system. In this analysis the spacecraft are assumed to be operating in deep space and the orbital motion is ignored. However, the spacecraft are not operating in a pure vacuum, but are in a space plasma environment of rarified charge particles. This will cause the electric field strength calculation to deviate from the standard inverse square of separation distance relationship. In particular, the influence of this plasma environment on the stability of the system is investigated.

The spinning 2-craft Coulomb tether concept has a direct application for interferometric sensor missions. Here the sensor measurement of discrete free-flying spacecraft are combined to yield an equivalent sensor measurement of a much larger base-line. With the spinning Coulomb tether concept the 2 craft are sweeping out circular paths as they complete a revolution. If the target is not moving fast relative to the Coulomb tether rotation speed, then these measurements can be used for interferometric sensing. As illustrated in Figure 1, if the 2 spacecraft have unequal masses, then the inertial trajectories will be 2 circles of different radii, sweeping out 2 different disks in one revolution. The spinning Coulomb tether could be deployed away from Earth on a heliocentric orbit to search for near Earth asteroids. The relatively short separation distance of less than 100 meters would provide a wide field-of-view sensor which could look for unknown satellites. Further, the focal length of this aperture dish could be varied by changing the spacecraft charge and resulting in larger or smaller relative orbits. Another application involves a deep space mother ship deploying a free-flying sensor. The spinning Coulomb tether concept could provide a purely electrostatic means of keeping this sensor flying about the mother craft while taking images or other sensor measurement.

This paper is organized as follows. First the basic charged relative equations of motion are presented for a body in a space plasma environment, and their limitations are discussed. Then the stability of the spinning 2-craft Coulomb tether concept is investigated for a circular relative equilibria. Finally, open-loop charge maneuvers are investigated which allow the circular orbit radii to be changed over time. Numerical simulation illustrate the resulting performance.

2. Problem Statement

This spinning 2-craft Coulomb tether study assumes that the spacecraft are flying in deep space and are not orbiting any celestial body. Figure 2 shows an inertial frame

$\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ with its origin at the inertial cluster center of mass. Let $\boldsymbol{r}_i$ be the inertial position vector of the i^{th} spacecraft, q_i the spacecraft charge, and m_i the mass. The equations of motion are then given by

$$m_1\ddot{\boldsymbol{r}}_1 = \quad k_c \frac{q_1 q_2}{d^2} \mathrm{e}^{-d/\lambda_d} \hat{\boldsymbol{\imath}}_r \tag{1a}$$

$$m_2\ddot{\boldsymbol{r}}_2 = -k_c \frac{q_1 q_2}{d^2} \mathrm{e}^{-d/\lambda_d} \hat{\boldsymbol{\imath}}_r \tag{1b}$$

where $d = r_1 + r_2$ and $r_i = |\boldsymbol{r}_i|$, while $\hat{\boldsymbol{\imath}}_r = \boldsymbol{r}_1/r_1$ is the unit direction vector of craft 1. The parameter $k_c = 8.99 \cdot 10^9$ Nm^2/C^2 is the Coulomb constant. Due to the spacecraft flying in a space plasma environment, the typical $1/d^2$ electrostatic force magnitude function is modified with the exponential term. The additional drop off depends on the plasma Debye length parameter λ_d [10]. A charged spacecraft in a plasma will statistically attract more plasma particles with opposite charge to its own. A second craft a distance d apart would not only experience the charge of the first craft, but also this opposite charge gathering around it. In essence, this effect causes the first spacecraft charge to be shielded from the second craft. The stronger the shielding is, the shorter the Debye length λ_d. In low Earth orbits the Debye length is on the order of millimeters to centimeters, far too small for the Coulomb thrusting concept to be practical. However, at GEO the plasma is hotter and less dense which increases the Debye length to values of 100-1000 meters [1], [11]. Such high Earth orbits has been the typical flight regime of most Coulomb thrusting research. In deep space at 1 AU the Debye length reduces again due to the colder plasma and can range between 20-40 meters [1]. Note that once the spacecraft are more than 1-2 Debye lengths apart, the exponential drop off dominates which makes the Coulomb force ineffective.

Let τ me the mass ratio and be defined as

$$\tau = \frac{m_1}{m_2} \tag{2}$$

Because the inertial $\mathcal{N}$ frame origin is defined to be the cluster center of mass, the motion of the second satellite can be determined through

$$\boldsymbol{r}_2 = -\tau \boldsymbol{r}_1 \tag{3}$$

Using $\boldsymbol{r}_1 = r_1\hat{\boldsymbol{\imath}}_r$, $d = r_1(1+\tau)$, the equations of motion of the first craft can be written in the form

$$\ddot{\boldsymbol{r}}_1 = \frac{k_c}{m_1}\frac{q_1 q_2}{(1+\tau)^2} \mathrm{e}^{-r_1(1+\tau)/\lambda_d} \frac{\boldsymbol{r}_1}{r_1^3} \tag{4}$$

Next, let us define the effective gravitational parameter μ_1 as

$$\mu_1(r_1) = \mu_0 \mathrm{e}^{-r_1(1+\tau)/\lambda_d} \tag{5}$$

where

$$\mu_0 = -\frac{k_c}{m_1}\frac{q_1 q_2}{(1+\tau)^2} \tag{6}$$

is a constant, positive parameter due to $q_1 q_2 < 0$, then the charged spacecraft equations of motion can be written as

$$\ddot{\boldsymbol{r}}_1 = -\frac{\mu_1(r_1)}{r_1^3}\boldsymbol{r}_1 \tag{7}$$

If μ_1 is a constant, then these equations are equivalent to the equations of motion of the gravitational 2 body problem where μ would be the gravitational constant. Analogously, the equations of motion of the second satellite can be written as

$$\ddot{\boldsymbol{r}}_2 = -\frac{\mu_2(r_2)}{r_2^3}\boldsymbol{r}_2 \tag{8}$$

where $\boldsymbol{r}_2 = -r_2\hat{\boldsymbol{\imath}}_r$ and

$$\mu_2(r_1) = -\frac{k_c}{m_2}\frac{q_1 q_2}{(1+\tau)^2}\tau^2 \mathrm{e}^{-r_1(1+\tau)/\lambda_d} \tag{9}$$

For μ_1 to be constant in Eq. (5), the radius r_1 must be constant or the Debye length must be infinitely large. This dictates that the constant μ_1 scenario is only possible with circular relative orbits if the Debye length is not much larger than the separation distance. However, if $d \ll \lambda_d$, then μ_1 is a constant and all possible spacecraft trajectories must be conic solutions (i.e. circles, ellipses, parabolas or hyperbolas). However, it should be noted that the charged spacecraft motion can yield both a positive or negative effective gravitational parameter μ, resulting in either attractive or repulsive inter-spacecraft forces. As discussed in [7], the negative μ_1 case always results in unbounded hyperbolic motion about the unoccupied focus. The positive μ_1 case yields equivalent orbit shapes to the gravitational 2-body problem. For the Coulomb tether problem in this paper, the charge product $q_1 q_2$ is assumed to be negative.

Note that as with the gravitational 2-body problem, if the equations of motion are written of one satellite relative to another, the same vector equation is obtain as in Eq. (7), but with a different μ definition. If both case the solutions are conic sections for the large Debye length situation.

From a stability point of view, if $\lambda_d \gg d$ and μ_1 is a constant, then all charged relative trajectories will be stable. This scenario yields a Hamiltonian system with only conservative forces acting on it. The response is now equivalent to the motion of satellites about a planet whose orbits are stable.

Of interest is what occurs when the plasma Debye length is not ignorable and the Coulomb force is no longer modeled through the vacuum electrostatic potential function $-k_c q_1 q_2/r$. Let us consider the inertial angular momentum vector of craft 1 about the center of mass:

$$\boldsymbol{H}_1 = \boldsymbol{r}_1 \times m_1\dot{\boldsymbol{r}}_1 \tag{10}$$

Taking the inertial time derivative yields

$$\dot{\boldsymbol{H}}_1 = \boldsymbol{r}_1 \times m_1\ddot{\boldsymbol{r}}_1 = 0 \tag{11}$$

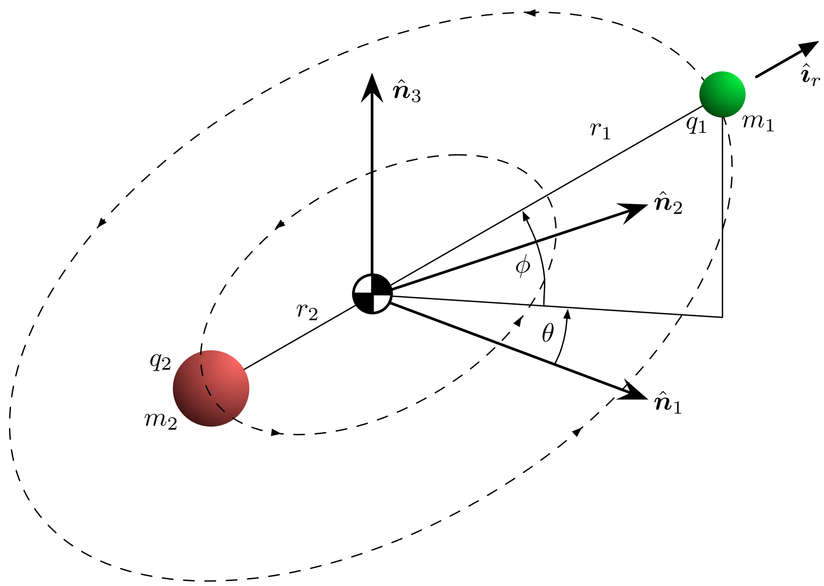


Figure 2. Illustration of coordinates used to describe spinning charged 2-craft cluster.

because the Coulomb force is always aligned with $\boldsymbol{r}_1$ regardless of the Debye length. Thus, whatever plane the initial position and velocity vectors form, all resulting motion will be within this plane even if the plasma charge shielding effect is considered.

The open loop charge product q_1q_2 required to maintain a circular orbit of radius r_c and velocity v_c is found as follows. Equating the centripetal acceleration magnitude with the inertial acceleration magnitude in Eq. (1) yields

$$\frac{v_c^2}{r_c} = -\frac{k_c}{m_1}\frac{q_1q_2}{r_c^2(1+\tau)^2}\mathrm{e}^{-r_1(1+\tau)/\lambda_d} \tag{12}$$

Solving for q_1q_2 we find the open-loop charge solution which will maintain a circular trajectory to be:

$$q_1q_2 = -v_c^2 r_c \frac{m_1}{k_c}(1+\tau)^2\mathrm{e}^{r_1(1+\tau)/\lambda_d} \tag{13}$$

This paper investigates the stability of this circular spinning Coulomb tether in the presence of the plasma shielding effect.

3. Stability Analysis

Reduced Equations of Motion

Before investigating the stability of the circularly-restricted Coulomb tether, let us reduce the equations of motion to a more convenient form using the spherical position coordinates (r, θ, ϕ) illustrated in Figure 2. This leads to conditions for a relative equilibria.

The system has six degrees of freedom which are the positions of the craft in space. However, setting the origin of the coordinate system at the center of mass of the formation, one only needs to consider the motion of one of the craft. The second spacecraft spherical coordinates are related to the first craft coordinates through:

$$r_2 = \tau r_1 \tag{14a}$$
$$\theta_2 = \pi + \theta_1 \tag{14b}$$
$$\phi_2 = -\phi_1 \tag{14c}$$

We will work in terms of the coordinates r_1, θ_1 and ϕ_1. From here on we rename these coordinates r, θ, and ϕ as shown in Figure 2. In terms of these spherical coordinates, the inertial kinetic energy of craft 1 is given by

$$K(r,\dot{r},\theta,\dot{\theta},\phi,\dot{\phi}) = \frac{m_1}{2}\left(\dot{r}^2 + r^2\cos^2\!\phi\,\dot{\theta}^2 + r^2\dot{\phi}^2\right) \tag{15}$$

The Lagrangian for this system is simply the kinetic energy:

$$L(r,\dot{r},\theta,\dot{\theta},\phi,\dot{\phi}) = K(r,\dot{r},\dot{\theta},\phi,\dot{\phi}) \tag{16}$$

The Lagrange d'Alembert principle:

$$\delta\int L(r,\dot{r},\theta,\dot{\theta},\phi,\dot{\phi})\mathrm{d}t + \int \boldsymbol{F}(r,\dot{r},\theta,\dot{\theta},\phi,\dot{\phi})\cdot(\delta r,\delta\theta,\delta\phi)\mathrm{d}t = 0 \tag{17}$$

can be used to derive the equations of motion, where $\boldsymbol{F}$ is the vector of generalized forces acting on the system. In the present case the generalized force is given by

$$\boldsymbol{F}(r,\dot{r},\theta,\dot{\theta},\phi,\dot{\phi}) = \left(\frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}}}{r^2(1+\tau)^2}, 0, 0\right) \tag{18}$$

The Lagrange d'Alembert principle then gives the following equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}}}{r^2(1+\tau)^2} \tag{19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \tag{19b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{19c}$$

We observe that the Lagrangian is independent of θ, which is therefore a cyclic variable. Associated with the cyclic variable θ is a conserved quantity, namely the angular momentum about the axis perpendicular to the plane of the orbit of the spacecraft pair. While the kinetic energy is not conserved for the general Debye length case, the cluster angular momentum is conserved because the Coulomb force is an internal force in all cases. Carrying out a Routhian reduction [12], we can use conservation of this quantity to derive the equations of motion, which are going to be independent of θ. Note that the generalized angular momentum is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \cos^2\!\phi\, \dot{\theta} \tag{20}$$

which is conserved according to Eq. (19b). For this system p_θ is the $\hat{\boldsymbol{n}}_3$ vector component of the inertial angular momentum vector $\boldsymbol{H}_1 = (H_1, H_2, H_3)$ in Eq. (10). Thus, given the initial conditions we can set $H_3 = p_\theta$. If $\phi = 0^{\circ}$ then p_θ is the angular momentum magnitude. Using Eq. (20) we are able to express $\dot{\theta}$ in terms of the radius r as

$$\dot{\theta} = \frac{H_3}{m_1 r^2 \cos^2\!\phi} \tag{21}$$

The Routhian is given

$$R(r, \dot{r}, \phi, \dot{\phi}) = \left[L - H_3 \dot{\theta}\right]_{\dot{\theta} = \frac{\eta}{m_1 r^2 \cos^2\phi}}$$
$$= \frac{1}{2}\left(m_1 \dot{r}^2 + m_1 r^2 \dot{\phi}^2 - \frac{H_3^2}{m_1 r^2 \cos^2 \phi}\right) \tag{22}$$

The reduced equations of motion are then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = \frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}}}{r^2(1+\tau)^2} \tag{23a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{\phi}}\right) - \frac{\partial R}{\partial \phi} = 0 \tag{23b}$$

which result in

$$m_1 \ddot{r} - m_1 r \dot{\phi}^2 - \frac{H_3^2}{m_1 r^3 \cos^2 \phi} = \frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}}}{r^2(1+\tau)^2} \tag{24a}$$

$$m_1 r^2 \ddot{\phi} + 2 m_1 r \dot{r} \dot{\phi} + \frac{H_3^2 \tan\phi}{m_1 r^2 \cos^2 \phi} = 0 \tag{24b}$$

For the following stability analysis the lack of θ in the reduced equations of motion provides an ideal simplification. For the un-perturbed orbit, θ is the in-plane angular position of the spacecraft. If a perturbed orbit has a slightly different $\dot{\theta}$ rate, or orbit period, then this neighboring trajectory is still considered stable in the orbital sense.

Let our circular orbit correspond to $r = r_c > 0$ (a constant), $\dot{r} = 0$, $\phi = \phi_c = 0$ and $\dot{\phi} = 0$. The generalized angular momentum corresponding to this orbit is given by

$$H_3^2 = -\frac{m_1 r_c k_c q_1 q_2 \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}}}{(1+\tau)^2} \tag{25}$$

We immediately note that for a bounded circular orbit, we require that

$$q_1 q_2 \leq 0 \tag{26}$$

Linearizing equations (24) about the nominal circular orbit, we obtain

$$m_1 \delta\ddot{r} + \frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}}}{r_c^2 \lambda_d (1+\tau)^2}\left(1 + \tau - \frac{\lambda_d}{r_c}\right)\delta r = 0 \tag{27a}$$

$$m_1 r_c^2 \delta\ddot{\phi} - \frac{k_c q_1 q_2 \exp^{-\frac{r_c(1+\tau)}{\lambda_d}}}{r_c(1+\tau)^2}\delta\phi = 0 \tag{27b}$$

where we have substituted H_3^2 by the expression in Eq. (25).

Note that the linearized equations are decoupled. Regarding the linearized radial equation of motion, the eigenvalues are given by

$$s_r = \pm \frac{H_3}{m_1 r_c^2}\sqrt{\frac{1}{\lambda_d}\left(1 + \tau - \frac{\lambda_d}{r_c}\right)} \tag{28a}$$

$$= \pm\sqrt{-\frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}}}{m_1 r_c^3 \lambda_d (1+\tau)^2}\left(1 + \tau - \frac{\lambda_d}{r_c}\right)} \tag{28b}$$

Noting that $q_1 q_2 < 0$, one eigenvalue will have a real positive value if

$$\lambda_d \leq r_c(1+\tau) \tag{29}$$

Thus, if the spacecraft separation distance $d = r_c(1+\tau)$ is greater than the Debye length λ_d, the circularly spinning Coulomb tether with constant charges is guaranteed to be unstable. For small separation distances where $d < \lambda_d$ the eigenvalues are purely imaginary, providing only marginal stability of the linearized in-plane motion.

Regarding the out of plane motion, the linearized ϕ equation has the following eigenvalues

$$s_\phi = \pm\imath \frac{H_3}{m_1 r_c^2} \tag{30a}$$

$$= \pm\imath\sqrt{-\frac{k_c q_1 q_2 \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}}}{m_1 r_c^3 (1+\tau)^2}} \tag{30b}$$

Because $q_1 q_2 < 0$, we immediately see that the linearized ϕ equation is marginally stable regardless of the Debye length value. However, in this case it can be argued that this linearized stability result does yield a stable equilibrium motion for the nonlinear system. Recall that the angular momentum vector $\boldsymbol{H}_1$ is conserved, which led to the argument that the spinning 2-craft Coulomb tether motion will always be planar. If the orbit plane is not in the nominal $(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2)$ plane, but rather is inclined by an angle i, this angle is determined through the initial $\boldsymbol{r}_1(t_0)$ and $\dot{\boldsymbol{r}}_1(t_0)$ vectors. The spherical coordinate ϕ will then be bounded by this constant inclination angle.

$$|\phi(t)| \leq i \tag{31}$$

Thus, regardless of initial conditions, the out-of-plane angle $\phi(t)$ is guaranteed to be stable for the nonlinear system thanks to the conservation of angular momentum in the presence of plasma charge shielding.

To investigate the nonlinear stability of the radial motion we can consider only the reduced radial equations of motion in Eq. (24a) for the planar motion without loss of generality. The nominal circular orbit angular momentum H in is written using Eq. (6) as

$$H^2 = \mu_0 m_1^2 r_c \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}} \tag{32}$$

Using $H_3 = H$ and $\phi = 0$ for the planar case, the radial equations of motion in Eq. (24a) are reduced to the form

$$\ddot{r} + F(r) = 0 \tag{33}$$

with

$$F(r) = \frac{\mu_0}{r^3}\left(r\mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}} - r_c \mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}} \right) \tag{34}$$

where r_c is the nominal circular orbit radius. Note that the angular momentum expression has been absorbed into the equivalent gravitational constant μ_0 in this formulation. If $r = r_c$, then $\ddot{r} = 0$ and the circular reference motion is retained.

Because the radial acceleration only depends on the radius r and not $\dot{r}$, the $F(r)$ function can be written as the gradient of the potential function $V_F(r)$

$$V_F(r) = -\frac{\mu_0}{r}\mathrm{e}^{-\frac{r(1+\tau)}{\lambda_d}} + \frac{\mu_0}{2r}\frac{r_c}{r}\mathrm{e}^{-\frac{r_c(1+\tau)}{\lambda_d}} - \frac{\mu_0(1+\tau)}{\lambda_d}\int_{-\frac{r(1+\tau)}{\lambda_d}}^{\infty} \frac{\mathrm{e}^{-s}}{s}\mathrm{d}s \tag{35}$$

Using $\ddot{r}_c = 0$, the equation of motion of radial deviations $\delta r = r - \delta r_c$ is then

$$\delta\ddot{r} = -\nabla_r V_F(r) \tag{36}$$

This potential function can be approximated about $r = r_c$ through the Taylor series expansion:

$$V_F(r + \delta r) = V_F(r_c) + k_1 \delta r^2 + k_2 \delta r^3 + \cdots \tag{37}$$

where

$$k_1 = \frac{\mu_0}{2r_c^3}\left(1 - \frac{d}{\lambda_d}\right)\mathrm{e}^{-\frac{d}{\lambda_d}} \tag{38}$$

$$k_2 = \frac{k_1}{3r_c}\left(\frac{d^2}{\lambda_d^2} + 4\frac{d}{\lambda_d} - 6\right) \tag{39}$$

with $d = r_c(1+\tau)$ being the spacecraft separation distance. For the case where $d < \lambda_d$ (the marginally stable linearized result) the quadratic term has $k_1 > 0$. The potential function V_F is thus guaranteed to have a finite neighborhood about $\delta r = 0$ for which V_F is positive definite in δr.

The Lagrange-Dirichlet stability states that an equilibrium point is stable if the second derivative of the potential function is positive definite [13], [14]. Using the potential function expansion in Eq. (37) it is evident that

$$\left.\frac{\mathrm{d}^2 V_F}{\mathrm{d}r^2}\right|_{r=r_c} = k_1 + \mathcal{O}(\delta r) > 0 \tag{40}$$

for some finite neighborhood about the origin.

Alternatively, we can define the candidate Lyapunov function V to study the nonlinear stability of $r(t)$.

$$V(\delta r, \delta\dot{r}) = \frac{1}{2}\delta\dot{r}^2 + V_F(r_c + \delta r) - V_F(r_c) \tag{41}$$

Note that at the equilibrium states $\delta r = \delta\dot{r} = 0$ that $V(\delta r) = 0$. Further, because there exists a neighborhood about the equilibrium where V_F is positive definite, this Lyapunov function is also locally positive definite. Evaluating the Lyapunov rate $\dot{V}$ yields

$$\dot{V} = \delta\dot{r}\left(\delta\ddot{r} + \nabla_r V_F\right) = 0 \tag{42}$$

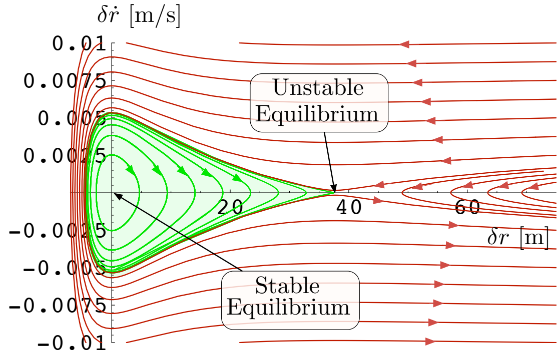
Because $\dot{V} \leq 0$ the nonlinear radial motion is guaranteed to be locally stable about the equilibrium.

Table 1. Numerical Simulation Parameters

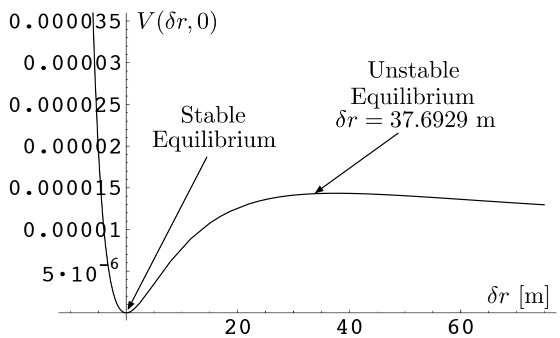
Parameter	Value	Units
m_1/m_2	50/75	kg
k_c	8.99×10^9	$\frac{\mathrm{Nm}^2}{\mathrm{C}^2}$
q_1/q_2	10/-10	μC
r_{c1}/r_{c2}	15/10	m
v_{c1}/v_{c2}	11.119/-7.413	mm/s
$\delta r_1(t_0)$	0.3	m
$\theta(t_0)$	0.0	deg
$\phi(t_0)$	1	deg

Numerical Simulation

To illustrate the stability of a spinning 2-craft Coulomb tether in deep space for different Debye length cases, the following numerical simulations are performed. The spacecraft masses, charges, and other relevant simulation parameters are listed in

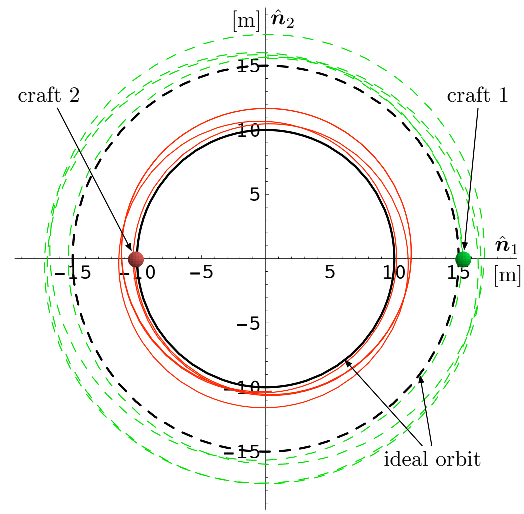


(a)Phase Portrait for $(\delta r, \delta\dot{r})$ Constant Angular Momentum.

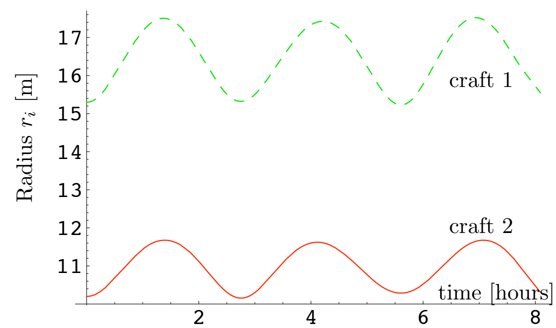


(b)Lyapunov Function for Ranges of δr Values.

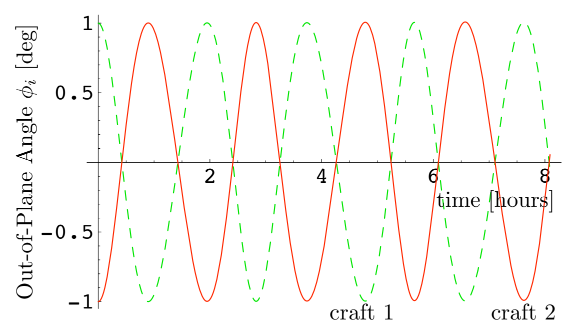
Figure 3. Stability Illustrations of Numerical Simulation.



(a)Planar Projection of Motion.



(b)Spacecraft Radii



(c)Out-of-Plane Motion

Figure 4. Numerical simulation results for stable case 1 with a large Debye length of λ_d = 50 meters for craft 1 (- - -) and craft 2 (——). The unperturbed motion is shown in black.

Table 1. Note that $m_2 > m_1$ to have both craft travel circular trajectories of different radii. With the given velocities it will take about 2.35 hours for the nominally circular Coulomb tether to complete one revolution. With higher charge limits this period could be reduced.

First let us investigate the phase plot of this nominally circular Coulomb tether with simulation parameters as specified in Table 1. The basic equations of motion in Eq. (1) are integrated for a range of initial δr and $\delta\dot{r}$ perturbations while holding the angular momentum magnitude H and spacecraft charges constant. The charges q_1 and q_2, and thus the parameter μ_0, are computed for the nominally circular trajectory were craft 1 has a circular radius of r_{c1} and speed v_{c1}, while craft 2 has a radius r_{c2} and speed v_{c2}. The Debye length is set to $\lambda = 50$ meters, for which the stability condition $d < \lambda_d$ is satisfied. The resulting phase portrait is shown in Figure 3(a). The teardrop region around the origin results in stable motions about this equilibrium as predicted. Figure 3(b) illustrates the position dependent behavior of the Lyapunov function V. About the origin $V(\delta r, 0)$ is locally positive definite. However, it is interesting to note that for a given H and μ_0 value, there exists a second circular equilibrium point. For this setup this equilibrium is hyperbolic, resulting in unstable motion. This would be expected because here the separation distance has grown larger than the Debye length.

Next the three-dimensional charged spacecraft motion is considered for two different Debye length cases. In case 1 the Debye length is 50 meters, which is larger than the 25 meter nominal separation distance. Case 2 sets λ_d = 20 meters which should result in unstable motion. Both cases use the same radial distance error δr and out of plane motion ϕ shown in Table 1. The initial velocity vectors were not perturbed in these simulations, only the initial positions. As a result, the perturbed Coulomb tether has an orbit plane inclination of $i = \phi(t_0) = 1^{\text{o}}$.

The inertial nonlinear equations of motion in Eq. (1) are integrated for 5 nominal orbit periods in this numerical simulation. The resulting motion for case 1 is illustrated in Figure 4. The planar projection of the 2 craft trajectories in Figure 4(a) shows the initial positions of the craft as small spheres, the nominal circular trajectories as black lines. The actual perturbed motion oscillates about the nominally circular trajectories, but does not form closed curves. If the Debye length were negligible here than the perturbed motion would also form conic solutions. However, with the plasma shielding active the inverse square Coulomb force is weakened further as the separation distances increase. The radial and out-of-plane motion coordinates are illustrated in Figures 4(b) and 4(c). As predicted, the angle ϕ is bounded in magnitude by the perturbed orbit inclination angle of 1 degree.

If the Debye length is reduced to 20 meters in case 2, then the numerical simulation yields unstable relative trajectories as illustrated in Figure 5. With the same initial perturbation, the plasma shielding effect is sufficient to destabilize the relative motion and cause the radial separation distance to grow infinitely large. The angular out-of-plane coordinate ϕ remains bounded by the orbit inclination angle and reduces in value as the spacecraft separate.

4. Coulomb-Tether Length Reconfiguration

One benefit of the Coulomb tether concept is that the spacecraft charges q_i can be regulated to desired values. For example, if the charges are lowered, then the effective gravitation parameter μ is reduced and the craft would increase their separation distance. This section outlines a method to reconfigure a spinning Coulomb tether and change the circular orbit radius to a new value.

For the case where the Debye length can be ignored ($d \ll \lambda_d$), the equations of motion of spacecraft 1 relative to the cluster center of mass as given by

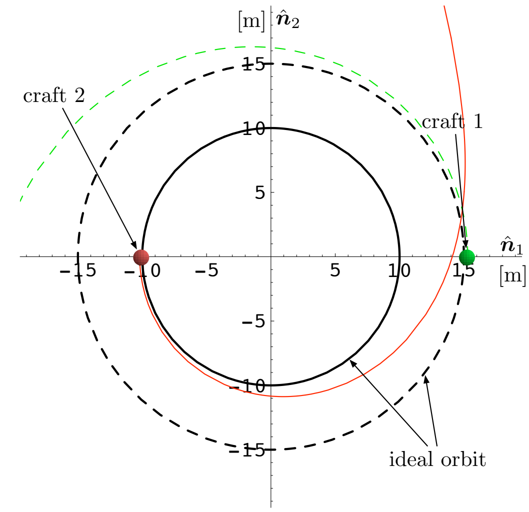
$$\ddot{\boldsymbol{r}}_1 = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1 \tag{43}$$

where the effective gravitational parameter μ_1 is the constant

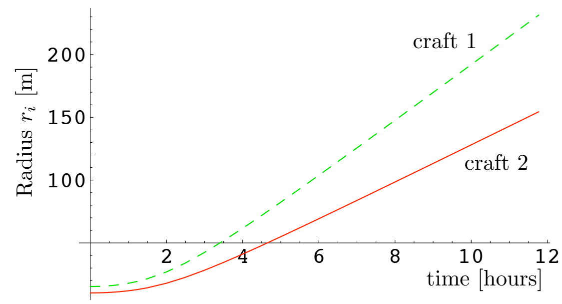
$$\mu_1 = -\frac{k_c}{m_1}\frac{Q_{12}}{(1+\tau)^2} \tag{44}$$

If $Q_{12} = q_1 q_2 < 0$, the $\mu_1 > 0$ and a gravity-like attractive force is experienced between the two spacecraft. This scenario allows us to be motivated by the gravitational orbit boost maneuvers such as the Hohmann transfer [15] to develop a spinning Coulomb tether reconfiguration maneuver. Following equivalent steps as are used to derive the energy equation for the gravitational 2-body problem [16], an equivalent energy equation can be found for the charged spacecraft motion.

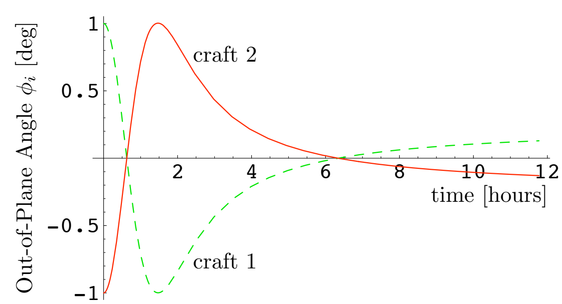
$$\frac{v_1^2}{2} - \frac{\mu_1}{r_1} = -\frac{\mu_1}{2a} \tag{45}$$



(a)Planar Projection of Motion.



(b)Spacecraft Radii



(c)Out-of-Plane Motion

Figure 5. Numerical simulation results for unstable case 2 with a small Debye length of λ_d = 20 meters for craft 1 (- - -) and craft 2 (——). The unperturbed motion is shown in black.

Here r_1 and v_1 are the radius and speed of craft 1, and a is the semi-major axis of the resulting conic motion. This energy equation is very useful for the 2-craft Coulomb tether problem because it can be used to determine the how much speed the craft can have without resulting in unbounded parabolic ($a \to \infty$) or hyperbolic ($a < 0$) motion. For example, to achieve a parabolic trajectory with a given radius r_1 and charge product Q_{12}, the condition $v_1^2 = 2\mu_1$ must be true

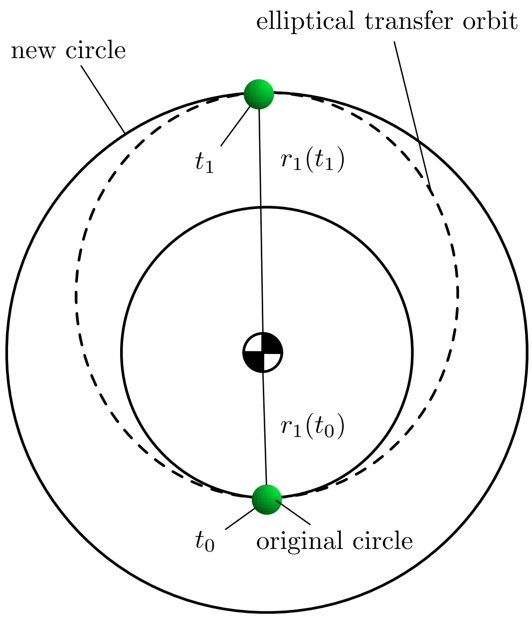


Figure 6. Illustration of the Coulomb Tether Reconfiguration Maneuver.

leading to an escape speed of

$$v_{1,\text{esc}} = -\frac{2}{r_1}\frac{k_c}{m_1}\frac{Q_{12}}{(1+\tau)^2} > 0 \tag{46}$$

For the gravitational 2-body problem the Hohmann transfers are obtain $\Delta \boldsymbol{v}$ maneuvers to transfer between 2 circular orbits using an elliptical transfer orbit. During the first burn the velocity magnitude is changed to make the satellite increase or decrease its radius. The burn is selected such that the next extremal point is at the desired orbit radius. After applying a second burn the desired circular orbit speed is maintained.

With the spinning 2-craft Coulomb tether problem it is not possible to apply an impulsive velocity change $\Delta \boldsymbol{v}$ using the Coulomb forces. Instead the effective gravitational parameter μ_1 is changed to weaken or strength the attractive Coulomb force. In this manner Coulomb tether reconfigurations are possible which are equivalent to the Hohmann gravitational maneuver, as illustrated in Figure 6.

Without loss of generality, let's assume we are going to increase our orbit radius of craft 1 from $r_0 = r_1(t_0)$ to $r_1(t_1) = \gamma r_1(t_0)$, where $\gamma > 0$ is a orbit radius scaling parameter. At time t_0 the gravity-like parameter μ_1 must be changed such that the given spacecraft speed is greater than the circular orbit speed and the craft fly apart. At time t_1 the craft has reached apoapses and the μ_1 parameter must be changed again to maintain the new circular orbit. Note that each μ_1 change is accomplished using Eq. (44) by changing the spacecraft charge levels.

Given the initial circular orbit of radius r_0 and speed v_0, the initial parameter $\mu_1(t_0^-)$ must be

$$\mu_1(t_0^-) = v_0^2 r_0 \tag{47}$$

The elliptical transfer orbit will have a semi-major axis of

$$a = \frac{r_0 + \gamma r_0}{2} = r_0 \frac{1+\gamma}{2} \tag{48}$$

To perform this maneuver we are changing the effective gravitational parameter μ_1 instantaneously such that the resulting elliptical motion will reach the desired final relative orbit altitude. The spacecraft charge can reach maximum values within milli-seconds, make the instantaneous charge change assumption valid. Thus, to write the energy equation at time t_0 where we enter the transfer orbit by changing $\mu_1(t_0^-)$ to $\mu_1(t_0)$, we still have a radius of r_0 and the initial circular orbit speed v_0.

$$\frac{v_0^2}{2} - \frac{\mu_1}{r_0} = -\frac{\mu_1}{2a} \tag{49}$$

Using the semi-major axis a in Eq. (48), this leads to the condition

$$v_0^2 r_0 = \mu_0 = \mu_1 \left(\frac{2\gamma}{1+\gamma}\right) \tag{50}$$

Because μ_1 is proportional to the charge product Q_{12}, we can state that to enter the desired transfer orbit at time t_0, the spacecraft charge product must be:

$$Q_{12}(t_0) = Q_{12}(t_0^-) \cdot \left(\frac{1+\gamma}{2\gamma}\right) \tag{51}$$

The percentage change is

$$\frac{Q_{12}(t_0) - Q_{12}(t_0^-)}{Q_{12}(t_0^-)} = \frac{1-\gamma}{2\gamma} \cdot 100\% \tag{52}$$

At time t_1 a new parameter $\mu_1(t_1)$ is needed to re-circularize the orbit. The spacecraft velocity at apoapses is $v_1 = v(t_1)$ and the radius is $r_1 = r_0\gamma$. To maintain a circular orbit, the effective gravitational parameter μ_1 must be changed from $\mu_1(t_0)$ to $\mu_1(t_1)$ such that

$$v_1^2 r_0 \gamma = \mu_1(t_1) \tag{53}$$

Expressing the transit orbit energy equation at apoapses we find

$$\frac{v_1^2}{2} - \frac{\mu_1(t_0)}{r_0\gamma} = -\frac{\mu_1(t_0)}{2a}$$

Substituting in the transfer orbit semi-major axis a and solving for $v_1^2 r_0 \gamma$ yields

$$v_1^2 r_0 \gamma = \mu_1(t_0)\left(2 - \frac{2}{1+\gamma}\right) = \mu_1(t_0)\frac{2}{1+\gamma} \tag{54}$$

Using Eq. (54) and (50) we find

$$\begin{aligned}\mu_1(t_1) &= \mu_1(t_0)\frac{2}{1+\gamma} = \mu_0(t_0^-)\frac{1+\gamma}{2\gamma}\frac{2}{1+\gamma} \\ &= \frac{\mu_0(t_0^-)}{\gamma} \end{aligned} \tag{55}$$

Thus, the final effective gravitational parameter is simply the initial value on the original circular orbit divided by the scaling factor γ. Finally, the spacecraft charge product Q_{12} at time t_1 must then be

$$Q_{12}(t_1) = Q_{12}(t_0^-)\frac{1}{\gamma} \tag{56}$$

The percent change with respect to the original $Q_{12}(t_0^-)$ is

$$\frac{Q_{12}(t_1) - Q_{12}(t_0^-)}{Q_{12}(t_0^-)} = \frac{1-\gamma}{\gamma}\cdot 100\% \tag{57}$$

The percent change with respect to the intermediate $Q_{12}(t_0)$ value is

$$\frac{Q_{12}(t_1) - Q_{12}(t_0)}{Q_{12}(t_0)} = \frac{1-\gamma}{1+\gamma}\cdot 100\% \tag{58}$$

Thus, by instantaneously change the spacecraft charge product using Eqs. (51) and (57), a Hohmann-transfer like reconfiguration of the circular spinning Coulomb tether is achieved.

5. Conclusions

The concept of a spinning 2-craft Coulomb tether is introduced in this paper. This is the first open-loop stable Coulomb spacecraft mission scenario that has been investigated. All previous work on virtual Coulomb structures, static nadir aligned Coulomb tethers, or general spacecraft cluster control required feedback control laws to stability the cluster shape, size and orientation. The analysis shows that the nonlinear radial motion is locally stable if the spacecraft separation distance is less than the Debye length, and it is guaranteed to be unstable if it is larger than the Debye length. The out-of-plane motion is shown to always be stable thanks to the conservation of angular momentum. This stability results are verified in two numerical simulations which illustrate both stable and unstable configurations. Further, open-loop charge maneuvers are illustrated which are inspired by the gravitational Hohmann transfer orbit problem. To reconfigure the separation distance of the Coulomb tether, instantaneous charge changes are computed which change the effective gravitational parameter and allow the tether length to expand or reduce to a desired value.

Acknowledgment

We would like to thank Professor Anthony M. Bloch from the University of Michigan, and Professor Christopher Hall from Virginia Tech, for several helpful conversations on the topic of this paper.

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