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SHADOW SET CONSIDERATIONS FOR MODIFIED RODRIGUES PARAMETER ATTITUDE FILTERING

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Rigid body attitude estimation algorithms have been previously formulated using Modified Rodrigues Parameter (MRP) attitude sets. MRP attitude estimation algorithms are attractive because they have been shown to have equal accuracy to and faster initial convergence than similar quaternion based filters and they avoid the quaternion constraint problem. These algorithms make use of the fact that MRP sets are not unique. Two possible MRP sets can describe a particular orientation, and singularity avoidance can be performed by switching between the original MRP set and the alternate set, known as the shadow set. Unfortunately, the non-uniqueness of MRPs can lead to significant attitude estimation errors through improper calculation of the measurement residual. The present work examines the details required for proper implementation of a MRP attitude estimation algorithm, specifically the technical details of when and how to switch to and from the MRP shadow set when calculating the measurement residual.

INTRODUCTION

Attitude estimation is often performed using an extended Kalman filter (EKF) with quaternions as the attitude measure.[1,2] Quaternions lend themselves well to attitude estimation as they represent a redundant, nonsingular attitude description with globally nonsingular kinematics, elegant successive rotation expressions, and rigorously linear kinematic differential equations. However, the quaternion unit norm constraint complicates matters and has led to extended discussions of attitude estimation and constraints.[3,4,5]

Usually, an error quaternion is estimated for each measurement assuming small angles such that a three-component representation may be used. This error quaternion is then combined with a quaternion propagated using state dynamics to arrive at a measurement updated attitude estimate. This method is used in both the Multiplicative EKF (MEKF), which is thoroughly discussed in Reference 6 and Reference 7, and the Additive EKF (AEKF).[8,9,10] A more recent approach, proposed by Zanetti, Majji, Bishop and Mortari, involves a Lagrange multiplier formulation to solve for all four components of the error quaternion.[11]

Other attitude parameterizations can be used in Kalman filtering assuming that appropriate strategies are employed to avoid singularities. Examples of other parameterizations include Euler angles,[12,13] Rodrigues parameters,[14] and Modified Rodrigues Parameters (MRPs). MRPs are of particular interest as they are a minimal three parameter attitude set which are nonsingular for any rotation other than multiples of 2π. Schaub and Junkins[15] note they are not unique; two MRP sets

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exist to describe a particular orientation, and the second set, known as the shadow set, is nonsingular for non-zero rotations. Therefore, singularity avoidance can be performed by switching between the two MRP sets. Further, both MRP sets obey the same differential equations making for easy implementation.

MRPs have been used to develop globally stabilizing feedback control,[16] optimal attitude control,[17] and sliding mode control for maneuvers.[18] They were first explored as an attitude estimation parameterization in 1996.[19] Lee and Alfriend present an additive divided difference filter using MRPs, but do not discuss the transformation of the covariance matrix when switching to the shadow MRP set.[20] Cheng and Crassidis propose using MRPs in a particle filter and they mention that MRP switching may cause discontinuities of the covariance, but do not provide an appropriate covariance mapping, although it is not actually required in their particle filtering approach.[21] Jizheng, Jianping, and Qun also note the covariance estimate experiences a discontinuity at the point where the MRP is switched to the shadow set and propose a first order covariance mapping.[22] Karlgaard and Schaub provide a first order covariance mapping for use in an EKF, and additionally provide first- and second-order transformations suitable for use in Divided Difference Filters (DD1, DD2).[23] Furthermore, they show an MRP EKF to have equal accuracy to and faster initial convergence than quaternion filters with slightly faster numerical evaluation and vastly simpler coding implementation.

The present work examines the details required for proper implementation of this algorithm, specifically the technical details of when and how to switch to and from the MRP shadow set. A review of MRPs is presented, followed by the derivation of an EKF which utilizes MRPs complete with an appropriate first-order analytical covariance mapping to be used when switching the MRPs to or from their shadow set. Next the technical issues of the non-uniqueness of MRPs with regard to the calculation of the measurement residual are examined. Finally, numerical simulation results demonstrating these issues and the performance of the MRP EKF are presented.

MODIFIED RODRIGUES PARAMETERS

The Modified Rodrigues Parameter vector $\boldsymbol{\sigma}$ is defined in terms of the principal rotation elements as

$$\boldsymbol{\sigma} = \hat{\boldsymbol{e}} \tan\left(\frac{\Phi}{4}\right) \tag{1}$$

where $\hat{\boldsymbol{e}}$ is the principal rotation axis, and Φ is the principal rotation angle.[15,24] By examining Equation (1) it can see that the MRP attitude description goes singular when $\Phi \to \pm 360\,\text{deg}$.

Because the sets $(\hat{\boldsymbol{e}}, \Phi)$ and $(\hat{\boldsymbol{e}}, \Phi')$, where $\Phi' = \Phi - 2\pi$, describe the same orientation the MRP shadow set can be derived as

$$\boldsymbol{\sigma}^S = -\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}^T\boldsymbol{\sigma}} = \hat{\boldsymbol{e}} \tan\left(\frac{\Phi - 2\pi}{4}\right). \tag{2}$$

Note that one set of MRPs corresponds to a principal rotation $\Phi \geq 180\,\text{deg}$ and the other to $\Phi \leq 180\,\text{deg}$, so while the original MRP set approaches the singularity the shadow set approaches zero. Thus, the singularity can easily be avoided by switching between the original and shadow MRP sets. While arbitrarily switching between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^S$ is mathematically valid, it can be seen that a computationally convenient switching point is when $\|\boldsymbol{\sigma}\| = 1$ which corresponds to $\Phi = 180\,\text{deg}$. In practice, the exact point when $\|\boldsymbol{\sigma}\| = 1$ does not need to be determined as the MRP attitude measure is still mathematically well defined for $\|\boldsymbol{\sigma}\| > 1$, instead simply perform switching when $\|\boldsymbol{\sigma}\| > 1$.

Both sets of MRPs satisfy the differential equation

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[\left(1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)[\boldsymbol{I}_{3\times3}] + 2\left[\boldsymbol{\sigma}\right]_\times + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} = \frac{1}{4}\left[B(\boldsymbol{\sigma})\right]\boldsymbol{\omega} \tag{3}$$

where $\boldsymbol{\omega}$ represents the body angular velocity and $[\boldsymbol{\sigma}]_\times$ is the skew-symmetric cross product matrix given by

$$[\boldsymbol{\sigma}]_\times = \begin{bmatrix} 0 & -\sigma_3 & \sigma_2 \\ \sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & \sigma_1 & 0 \end{bmatrix}.$$

MRP KALMAN FILTER FORMULATION

A common attitude estimation problem involves propagating the state dynamics using the inertial angular velocity vector, sensed via a rate gyroscope, and correcting that estimate using a direct measurement of the body's attitude, via a star tracker, three-axis magnetometer, or other generic attitude sensor.[1,2] The discrete-time attitude measurements are incorporated through the use of a continuous-discrete extended Kalman Filter, as defined in Reference 25.

A commonly used approximation for rate gyroscope measurements assumes the gyroscope dynamics follow Farrenkopf's approximation[26]

$$\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}_b - \boldsymbol{\eta}_\omega \tag{4}$$

$$\dot{\boldsymbol{\omega}}_b = \boldsymbol{\eta}_{\omega_b} \tag{5}$$

where $\tilde{\boldsymbol{\omega}}$ represents the sensed angular velocity, $\boldsymbol{\omega}$ the true angular velocity, $\boldsymbol{\omega}_b$ the measurement bias, and $\boldsymbol{\eta}_\omega$ and $\boldsymbol{\eta}_{\omega_b}$ unbiased uncorrelated random noise vectors. It follows that the state dynamics are given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\eta}) \tag{6}$$

where $\boldsymbol{x} = [\boldsymbol{\sigma}, \boldsymbol{\omega}_b]^T$ represents the state vector, $\boldsymbol{\eta} = [\boldsymbol{\eta}_\omega, \boldsymbol{\eta}_{\omega_b}]^T \sim N(0, \boldsymbol{Q})$ represents the process noise vector, and

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{4}\left[B(\boldsymbol{\sigma})\right](\tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}_b) \\ \boldsymbol{0}_{3\times3} \end{bmatrix} \tag{7}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\eta}) = \begin{bmatrix} -\frac{1}{4}\left[B(\boldsymbol{\sigma})\right]\boldsymbol{\eta}_\omega \\ \boldsymbol{\eta}_{\omega_b} \end{bmatrix}. \tag{8}$$

Thus, the continuous-time propagation equations are given by Equation (6) and the Lyapunov differential equation[25]

$$\dot{\boldsymbol{P}} = \boldsymbol{F}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{F}^T + \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}^T \tag{9}$$

where $\boldsymbol{P}$ represents the state covariance matrix, and

$$\boldsymbol{F} \equiv \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}} = \begin{bmatrix} \frac{1}{2}\left(\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\omega}}^T - \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\sigma}}^T - [\hat{\omega}]_\times + \hat{\boldsymbol{\sigma}}^T\hat{\boldsymbol{\omega}}\boldsymbol{I}\right) & -\frac{1}{4}\left[B(\hat{\boldsymbol{\sigma}})\right] \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \tag{10}$$

$$\boldsymbol{G} \equiv \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\eta}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}, \boldsymbol{\eta}=0} = \begin{bmatrix} -\frac{1}{4}\left[B(\hat{\boldsymbol{\sigma}})\right] & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \tag{11}$$

where $\hat{\boldsymbol{\sigma}}$ represents the current best estimate of the attitude MRP and $\hat{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}_b$ represents the current best estimate of the body angular velocity.

The attitude sensing device dynamics are assumed to take the form

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} - \delta\boldsymbol{\sigma} \tag{12}$$

where $\boldsymbol{\sigma}$ represents the true MRP attitude, $\tilde{\boldsymbol{\sigma}}$ represents the measured MRP, and $\delta\boldsymbol{\sigma} \sim N(0, \boldsymbol{R})$ is the measurement error, not the attitude estimation error. Using the measurement equation

$$\boldsymbol{h}(\boldsymbol{x}_k) = \hat{\boldsymbol{\sigma}}_k \tag{13}$$

these discrete-time measurements can be incorporated into the state estimate using the update equations given by

$$\hat{\boldsymbol{x}}_k^+ = \hat{\boldsymbol{x}}_k^- + \boldsymbol{K}_k \left[\tilde{\boldsymbol{\sigma}}_k - \hat{\boldsymbol{\sigma}}_k\right] \tag{14}$$

$$\boldsymbol{P}_k^+ = \left[\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k(\hat{\boldsymbol{x}}_k^-)\right] \boldsymbol{P}_k^- \left[\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k(\hat{\boldsymbol{x}}_k^-)\right]^T + \boldsymbol{K}_k \boldsymbol{R}_k \boldsymbol{K}_k^T \tag{15}$$

where $\hat{\boldsymbol{x}}_k^-$ and $\hat{\boldsymbol{x}}_k^+$ are the propagated and measurement corrected state estimates at time t_k, respectively, and $\boldsymbol{P}_k^-$ and $\boldsymbol{P}_k^+$ are the propagated and measurement corrected covariance estimates, respectively. Note the conventional Kalman filter covariance update equation is replaced with the Joseph formulation to improve numerical stability.[27] The Kalman gain matrix is given by

$$\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{H}_k^T(\hat{\boldsymbol{x}}_k^-) \left[\boldsymbol{H}_k(\hat{\boldsymbol{x}}_k^-) \boldsymbol{P}_k^- \boldsymbol{H}_k^T(\hat{\boldsymbol{x}}_k^-) + \boldsymbol{R}_k\right]^{-1} \tag{16}$$

where

$$\boldsymbol{H}_k(\hat{\boldsymbol{x}}_k^-) \equiv \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right|_{\hat{\boldsymbol{x}}_k^-} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix}. \tag{17}$$

If after propagating using Equation (6) or performing an update using Equations (14) and (15) $\|\boldsymbol{\sigma}\| > 1$ the MRP attitude set is switched to the shadow set. The shadow set transformation of the state vector is given by

$$\boldsymbol{x}^S = \begin{bmatrix} -\left(\boldsymbol{\sigma}^T \boldsymbol{\sigma}\right)^{-1} \boldsymbol{\sigma} \\ \boldsymbol{\omega}_b \end{bmatrix}. \tag{18}$$

Decomposing the covariance matrix into submatrices

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{\sigma\sigma} & \boldsymbol{P}_{\sigma\omega_b} \\ \boldsymbol{P}_{\sigma\omega_b}^T & \boldsymbol{P}_{\omega_b\omega_b} \end{bmatrix}$$

where $\boldsymbol{P}_{xx}$ is the covariance matrix of x and $\boldsymbol{P}_{xy}$ is the cross-correlation matrix between x and y, the mapping of the covariance matrix to the shadow set is given by[23]

$$\boldsymbol{P}^S = \begin{bmatrix} \boldsymbol{S}\boldsymbol{P}_{\sigma\sigma}\boldsymbol{S}^T & \boldsymbol{S}\boldsymbol{P}_{\sigma\omega_b} \\ \boldsymbol{P}_{\sigma\omega_b}^T \boldsymbol{S}^T & \boldsymbol{P}_{\omega_b\omega_b} \end{bmatrix} \tag{19}$$

where

$$\boldsymbol{S} = 2\sigma^{-4}\boldsymbol{\sigma}\boldsymbol{\sigma}^T - \sigma^{-2}\boldsymbol{I}_{3\times3}$$

and $\sigma^2 = \boldsymbol{\sigma}^T\boldsymbol{\sigma}$.

MRP SHADOW SET CONSIDERATIONS

Of particular interest here is the computation of the measurement residual $\boldsymbol{y}_k$, the difference between the measured $\tilde{\boldsymbol{\sigma}}_k$ and estimated attitude $\hat{\boldsymbol{\sigma}}_k$ at time t_k, which has not previously been discussed in detail.

$$\boldsymbol{y}_k = \tilde{\boldsymbol{\sigma}}_k - \hat{\boldsymbol{\sigma}}_k \tag{20}$$

Because the quantities $\tilde{\boldsymbol{\sigma}}_k$ and $\hat{\boldsymbol{\sigma}}_k$ represent MRP attitude descriptions it might appear Equation (20) should be evaluated using the MRP direct addition formula

$$\boldsymbol{y}_k = \frac{\left(1 - \hat{\boldsymbol{\sigma}}_k^T \hat{\boldsymbol{\sigma}}_k\right) \tilde{\boldsymbol{\sigma}}_k - \left(1 - \tilde{\boldsymbol{\sigma}}_k^T \tilde{\boldsymbol{\sigma}}_k\right) \hat{\boldsymbol{\sigma}}_k + 2\,[\tilde{\boldsymbol{\sigma}}_k]_\times \hat{\boldsymbol{\sigma}}_k}{1 + \tilde{\boldsymbol{\sigma}}_k^T \tilde{\boldsymbol{\sigma}}_k \hat{\boldsymbol{\sigma}}_k^T \hat{\boldsymbol{\sigma}}_k + 2 \hat{\boldsymbol{\sigma}}_k^T \tilde{\boldsymbol{\sigma}}_k} \tag{21}$$

or $[C(\boldsymbol{y}_k)] = [C(\tilde{\boldsymbol{\sigma}}_k)]\,[C(\hat{\boldsymbol{\sigma}}_k)]^T$ where

$$[C(\boldsymbol{\sigma})] = \boldsymbol{I}_{3\times 3} + \frac{8\,[\boldsymbol{\sigma}]_\times^2 - 4(1 - \boldsymbol{\sigma}^T \boldsymbol{\sigma})\,[\boldsymbol{\sigma}]_\times}{(1 + \boldsymbol{\sigma}^T \boldsymbol{\sigma})^2}.$$

This would yield the true attitude estimation error, not just the numerical difference of MRP values as in Equation (20). However, the Kalman filter is formulated using matrix math, and thus should be evaluated as such.

As discussed earlier, due to the non-uniqueness of MRPs, there are always two MRP sets to describe the same orientation. This can become an issue if $\|\tilde{\boldsymbol{\sigma}}\|$ or $\|\hat{\boldsymbol{\sigma}}\|$ is near 1.0. For example, if $\tilde{\boldsymbol{\sigma}} = [1, 0, 0]$ and $\hat{\boldsymbol{\sigma}} = [-1, 0, 0]$, which represents the same physical orientation as $\tilde{\boldsymbol{\sigma}}^S$, both $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ describe the same attitude and thus the measurement residual should be $[0, 0, 0]$. However, Equation (20) will result in a measurement residual of $[2, 0, 0]$ and the update equation given by Equation (14) will apply a correction when none is needed, thus degrading the estimate of the attitude.

To avoid this issue in practice, a new approach is proposed where the measurement residual is calculated a second time using

$$\boldsymbol{y}'_k = \tilde{\boldsymbol{\sigma}}_k^S - \hat{\boldsymbol{\sigma}}_k \tag{22}$$

where $\tilde{\boldsymbol{\sigma}}_k^S$ is evaluated using Equation (2). The value of $\boldsymbol{y}_k$ or $\boldsymbol{y}'_k$ with the smaller magnitude is then used in Equation (14) and estimation continues. Figure 1 illustrates graphically the situation where $\|\boldsymbol{y}'_k\| < \|\boldsymbol{y}_k\|$ and Algorithm 1 provides pseudocode for the proposed algorithm.

Algorithm 1 Proposed measurement residual algorithm.

$\boldsymbol{y}_k = \tilde{\boldsymbol{\sigma}}_k - \hat{\boldsymbol{\sigma}}_k$
if $\|\tilde{\boldsymbol{\sigma}}_k\| > \frac{1}{3}$ **then**
    $\boldsymbol{y}'_k = \tilde{\boldsymbol{\sigma}}_k^S - \hat{\boldsymbol{\sigma}}_k$
    if $\|\boldsymbol{y}'_k\| < \|\boldsymbol{y}_k\|$ **then**
        $\boldsymbol{y}_k = \boldsymbol{y}'_k$
    end if
end if

Performing this additional calculation at every time step does not represent a significant computational burden, however, an issue does develop when $\|\tilde{\boldsymbol{\sigma}}_k\| \to 0$. Here the shadow set $\|\tilde{\boldsymbol{\sigma}}_k^S\| \to \infty$ and is ill-defined. Note that in this scenario the magnitude of the original MRP set $\|\tilde{\boldsymbol{\sigma}}_k\|$ is always

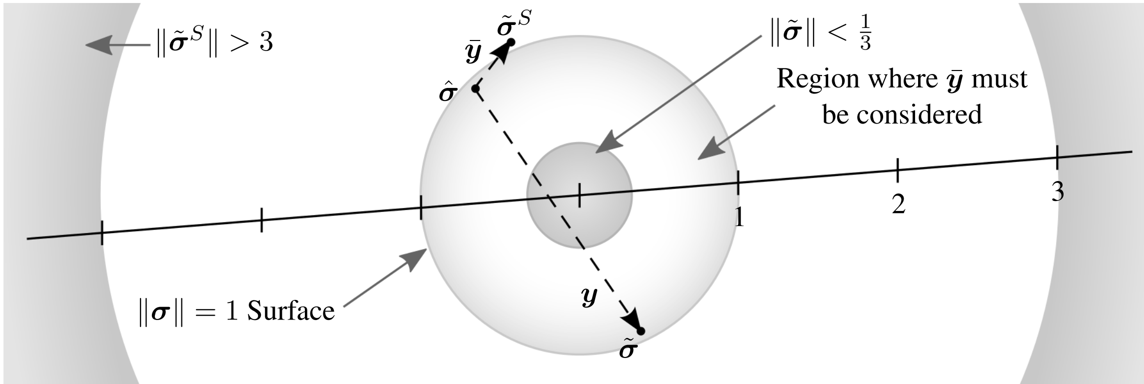


Figure 1: Illustration of possible measurement residual and region where $\boldsymbol{y}'$ must be considered.

less than the magnitude of the shadow MRP set $\|\tilde{\boldsymbol{\sigma}}_k^S\|$ and there is no need to evaluate Equation (22). For this reason a bound must be placed on when to evaluate Equation (22). Both $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ are always constrained to have a magnitude less than or equal to 1, which implies

$$\|\boldsymbol{y}_k\| \leq 2.$$

Therefore, if the magnitude of the measured MRP's shadow set $\tilde{\boldsymbol{\sigma}}_k^S$ is greater than 3 the value of $\boldsymbol{y}_k'$ must always be greater than $\boldsymbol{y}_k$

$$\|\tilde{\boldsymbol{\sigma}}_k^S\| > 3 \quad \longrightarrow \quad \|\boldsymbol{y}_k\| < \|\boldsymbol{y}_k'\|$$

and $\boldsymbol{y}_k'$ need not be calculated. By applying Equation (2) it is evident that

$$\|\tilde{\boldsymbol{\sigma}}_k^S\| > 3 \quad \longrightarrow \quad \|\tilde{\boldsymbol{\sigma}}_k\| < 1/3.$$

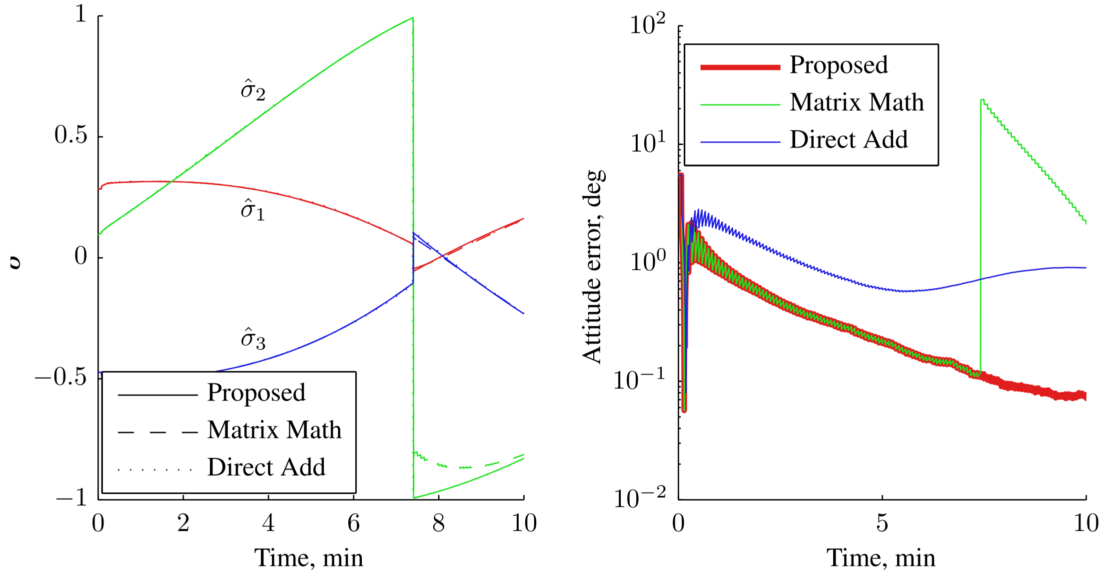
Thus, a conservative bound on when the calculation of $\boldsymbol{y}_k'$ can be ignored is when $\|\tilde{\boldsymbol{\sigma}}_k\| < 1/3$. Therefore, when $1/3 < \|\tilde{\boldsymbol{\sigma}}_k\| < 1$, as illustrated in Figure 1, the check described above should be computed.

RESULTS

A simple numerical simulation is presented here to illustrate the performance of the non-singular MRP EKF and highlight certain implementation details. In the following results the proposed method applies Algorithm 1 where two $\boldsymbol{y}_k$ values are computed. The matrix math method uses Equation (20) for a single $\boldsymbol{y}_k$ evaluation. Finally, the direct addition case applies Equation (21) to compute the true attitude estimate orientation error.

In this simulation the uncontrolled tumbling motion of a small spacecraft is modeled. The spacecraft is assumed to have principle inertia values of $I_1 = 4\,\text{kgm}^2$, $I_2 = 4\,\text{kgm}^2$, and $I_3 = 3\,\text{kgm}^2$. The initial attitude of the spacecraft is given by $\boldsymbol{\sigma}(t_0) = \begin{bmatrix}0.3 & 0.1 & -0.5\end{bmatrix}$. The initial angular velocity is given by $\boldsymbol{\omega}(t_0) = \begin{bmatrix}-0.2 & 0.2 & -0.192\end{bmatrix}$deg/s. These initial conditions are chosen as they quickly illustrate the issues with calculating the measurement residual as will be shown.

Attitude measurements are simulated at $0.2\,\text{Hz}$. These measurements are corrupted by white Gaussian noise with a standard deviation of $20\,\text{arcsec}$. Angular rate measurements are simulated at



(a) Various attitude estimates. (b) Attitude estimate error.

Figure 2: Results of simulation illustrating importance of checking $\boldsymbol{y}_k'$. The proposed method follows Algorithm 1 where two $\boldsymbol{y}_k$ values are computed. The matrix math method uses Equation (20) for a single $\boldsymbol{y}_k$ evaluation. The direct addition case uses Equation (21) to compute the true attitude estimate orientation error.

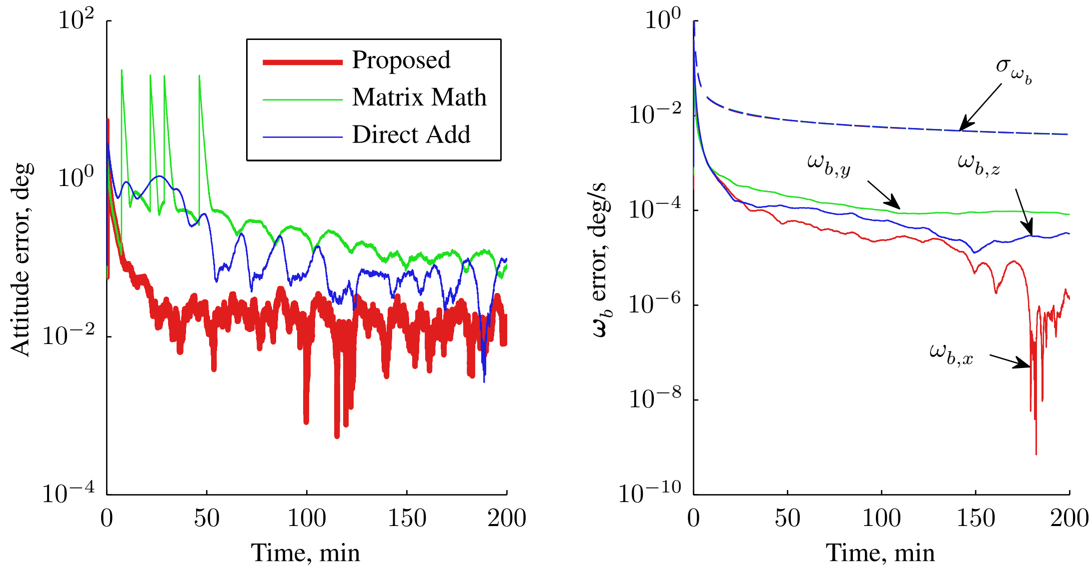
2.0 Hz, assuming a constant bias of $\boldsymbol{\omega}_b = \begin{bmatrix}-1.0 & 2.0 & -3.0\end{bmatrix}$ deg/hr and white Gaussian noise with a standard deviation of 0.001 deg/s..

The initial attitude estimate is $\hat{\boldsymbol{\sigma}} = \begin{bmatrix}0.0 & 0.0 & 0.0\end{bmatrix}$ and the initial angular rate bias estimate is $\hat{\boldsymbol{\omega}}_b = \begin{bmatrix}0.0 & 0.0 & 0.0\end{bmatrix}$. The initial variance of the attitude estimate is set to 0.175, the initial variance of the angular rate bias is set to $0.005\,\text{rad}^2/\text{s}^2$, and the initial covariances are set to zero. The angular rate process noise variance is set to $5 \times 10^{-5}\,\text{rad}^2/\text{s}^2$ and the angular rate bias process noise variance is set to $1 \times 10^{-16}\,\text{rad}^2/\text{s}^2$. The measurement error variance is set to $0.01\,\text{rad}^2/\text{s}^2$.

Figure 2 shows both the time history of the attitude estimate and the principal rotation error of the estimate for the first 10 min of the simulation. An example of when $\|\boldsymbol{y}_k'\| < \|\boldsymbol{y}_k\|$ can be seen at 7.42 min. The value of the measured and estimated MRP attitudes along with other relevant estimator quantities are listed in Table 1. As can be seen, simply calculating the vector difference

Table 1: Values of estimator quantities, and their magnitudes, at $t = 7.4$ min in the numerical simulation.

$\tilde{\boldsymbol{\sigma}}$	$\tilde{\boldsymbol{\sigma}}^S$	$\hat{\boldsymbol{\sigma}}$	$\boldsymbol{y}_k$	$\boldsymbol{y}_k'$
0.054 867	−0.054 890	−0.054 792	0.109 659	−0.000 087
0.993 141	−0.993 543	−0.992 450	1.985 591	−0.000 892
−0.101 273	0.101 314	0.101 665	−0.202 938	−0.000 371
0.999 797	1.000 203	0.999 147	1.998 945	0.000 970



(a) Attitude estimate error.

(b) Rate gyroscope bias estimate error using proposed method.

Figure 3: Results of numerical simulation showing convergence of estimate.

between $\tilde{\boldsymbol{\sigma}}_k$ and $\hat{\boldsymbol{\sigma}}_k$ results in a spuriously large error in the attitude estimate, whereas by using the shadow MRP set of the measured attitude the measurement residual is very close to zero.

The results of running the simulation for 200 min are shown in Figure 3. The rate gyroscope biases and their associated estimated covariances are shown in Figure 3b for the proposed method, and can be seen to quickly converge to the noise level. As shown in Figure 3a, the MRP EKF quickly converges to an error less than 1 deg in just over 1 min using the check in Equation (22) despite the relatively slow attitude measurement update rate. The same filter run without the $\boldsymbol{y}'_k$ check does converge, but takes much longer to do so and exhibits several large spikes with errors on the order of 20 deg. The applied 20 arcsec attitude measurement noise about all three axis corresponds to an attitude error of 0.038 deg which agrees well with the resulting attitude estimate.

CONCLUSIONS

The details associated with switching to the and from MRP shadow set in the context of a MRP based extended Kalman filter for attitude estimation are discussed. It is shown analytically and with numerical simulation that when calculating the measurement residual the fact that there are two valid MRP representation for any one attitude is important to remember and consider. Calculating the measurement residual using the attitude measurement MRP shadow set does not represent a significant computational burden; however, issues arise when $\|\tilde{\boldsymbol{\sigma}}\| \to 0$. A conservative bound of $\|\tilde{\boldsymbol{\sigma}}\| \geq 1/3$ has been established for when to check the value of $\boldsymbol{y}'$. The MRP EKF provides a globally nonsingular attitude estimation algorithm with a minimal attitude representation, but care must be taken when switching attitude estimate to and from the MRP shadow set.

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