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INITIAL CONDITIONS AND FUEL-OPTIMAL CONTROL FOR FORMATION FLYING SATELLITES

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INITIAL CONDITIONS AND FUEL-OPTIMAL CONTROL FOR FORMATION FLYING OF SATELLITES

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Abstract

This paper deals with the formulation and solution of the initial condition determination and fuel-optimal control problem in regard to formation flying of satellites. Unlike the relative station-keeping problem for satellites in the same plane, formation flying entails the creation of proper out-of-plane relative motion such that the satellites in the formation satisfy some mission requirements. Two methods for determining initial conditions of a satellite (Deputy), given the initial conditions of the chief satellite, for formation flying without thrust for a long period, are presented. The first method matches the mean J_2-induced angular drift rates of the two satellites, and the second method is based on imposing periodic boundary conditions on the relative position and velocity in a rotating coordinate system. A fuel-optimal impulsive thrusting scheme is developed to establish a formation. A fuel-optimal, low-thrust, variable Isp propulsion scheme is presented for orbit maintenance.

Introduction

Under ideal Two-Body assumptions, a satellite (Deputy) can be kept at a constant distance from another satellite (Chief) in a circular orbit, without the use of thrust, by choosing the right phasing and the inclination of the relative orbit. This type of a relative orbit has been proposed for the LISA[1] mission. Other proposed formation flying missions are ST3[2], ORION[3], Auroral Lites[4] , and Techsat-21[23]. Hill's equations[5] have been used to study relative motion of rendezvous mechanics. The key to establishing a formation using Hill's equations is to choose initial conditions that generate periodic solutions. Recently, Bond[6] developed an alternate set of equations which does not have the stability (secular drift) problem associated with Hill's equations. The shape of the projection of the relative orbit perpendicular to the radial (zenith-nadir) direction is of interest for the purpose of optical interferometry. Kong et al.[7] have studied various types of free and forced orbits, using Hill's equations, suitable for space based interferometes. Melton[8] presents a state transition matrix for relative motion between satellites in elliptic orbits in terms of a power series in eccentricity.

The attractive solutions to Hill's equations under ideal conditions get disturbed when perturbations due to the Earth's oblateness or aerodynamic drag are included in the model. The fuel consumption required for maintaining a formation to fight these perturbations will be prohibitive for more than a very short period of time. The primary perturbation of interest is due to J_2, which causes, among others, three important effects: Nodal regression, and drifts in perigee and the mean anomaly. These effects, using orbit-averaged quantities, are given below[9]:

$$\dot{\boldsymbol{\Omega}} = -1.5 J_2 \left(\frac{R_e}{p} \right)^2 n \cos i \tag{1}$$

$$\dot{\boldsymbol{\omega}} = 0.75 J_2 \left(\frac{R_e}{p} \right)^2 n \left(5 \cos^2 i - 1 \right) \tag{2}$$

$$\dot{M} = n + 0.75 J_2 \sqrt{1-e^2} \left(\frac{R_e}{p} \right)^2 n \left(3 \cos^2 i - 1 \right) \tag{3}$$

where Ω, is the longitude of the ascending node, ω, is the argument of perigee, M, is the mean anomaly, e, is the eccentricity, i, is the inclination, $p = a(1-e^2)$, $n = \sqrt{\mu / a^3}$, and a, is the semi-major axis.

If these effects are not controlled, the satellite formation will break down. Differential drag is not a major concern if the two satellites have similar aerodynamic characteristics. Kechichian[10] has developed a method for studying relative motion in the presence of oblateness and drag using a dragging and precessing reference frame. The formulation presented in this work is based on osculating elements rather than the mean elements.

Out-of-plane motion can be created using a node difference or an inclination difference between the satellites. It is much easier to deal with the node difference than the inclination difference because the latter gives rise to differential drift rates in Ω, ω, and M.

Schaub and Alfriend[11] developed an analytical solution to the initial condition problem by enforcing to first order, equal mean nodal rates and equal mean $\dot{\omega} + \dot{M}$. Unfortunately, the three equations, Eq. (1-3) cannot be satisfied simultaneously by two satellites with different mean a, e, and i. The mean elements are converted to corresponding osculating elements using a first order approximation taken from Brouwer[12].

In this paper, the two conditions of Ref. 11 are enforced numerically to determine the initial conditions of the deputy, given the initial conditions of the chief. An alternate constraint set is also developed in terms of the differences in energy and the polar component of the angular momentum, between the two satellites. A second technique is presented for the determination of initial conditions, using numerical integration of the differential equations subject to periodic boundary conditions on the relative position and velocity, as seen in a rotating coordinate system. This technique produces relative orbits with much smaller drift rates than the previous method but it is not convenient for use in the control problem.

Subsequently, the first technique for determining the initial conditions is utilized to develop a multi-impulse, minimum-fuel control scheme to establish the formation. A low-thrust, variable Isp, minimum-fuel control scheme is also developed for periodically correcting small errors in the formation due to other perturbations. This method of control can be implemented using plasma or ion propulsion.

Equations of Relative Motion

The equation of motion of a satellite under the influence of gravitational and thrust effects is given below:

$$\ddot{\mathbf{r}} = -\phi_r + \mathbf{u} \tag{4}$$

where, $\mathbf{r}$, is the position vector, ϕ, is the gravitational potential and ϕ_r is its gradient. The thrust acceleration is denoted by $\mathbf{u}$. The gravitational potential and its gradient[5], including the contribution of J_2, are shown below:

$$\phi = -\frac{\mu}{r}\left[1 - \frac{J_2}{2}\left(\frac{R_e^2}{r}\right)^2\left(3\frac{z^2}{r^2} - 1\right)\right] \tag{5}$$

$$\phi_r = \frac{\mu}{r^3}\mathbf{r} + \frac{J_2 \mu R_e^2}{2}\left[\frac{6z}{r^5}\hat{\mathbf{n}} + \left(\frac{3}{r^5} - \frac{15z^2}{r^7}\right)\mathbf{r}\right] \tag{6}$$

The expressions for the energy and the angular momentum are also useful.

$$E = \frac{1}{2}\mathbf{v}^T\mathbf{v} + \phi \tag{7}$$

$$\mathbf{H} = \mathbf{r} \times \mathbf{v} \tag{8}$$

It is well known that for the above model, the energy and the polar component of the angular momentum are conserved in the absence of thrust and drag**.**

Herein, variables with subscript 0 are used to denote the conditions of the chief. Any variable connected with the deputy is denoted by a subscript 1. The inertial relative displacement and velocity are defined as follows:

$$\delta\mathbf{r} = \mathbf{r_1} - \mathbf{r_0} \tag{9}$$

$$\delta\mathbf{v} = \mathbf{v_1} - \mathbf{v_0} \tag{10}$$

The relative motion between two satellites in general elliptic orbits is best visualized in a rotating frame constructed with the following coordinates:

$$\delta x = \frac{\delta\mathbf{r}^T\mathbf{r}_0}{r_0} \tag{11}$$

$$\delta y = \frac{\delta\mathbf{r}^T(\mathbf{H}_0 \times \mathbf{r}_0)}{|\mathbf{H}_0 \times \mathbf{r}_0|} \tag{12}$$

$$\delta z = \frac{\delta \mathbf{r}^T \mathbf{H}_0}{H_0} \quad (13)$$

The relative velocities in this rotating coordinate system are given by the following:

$$\delta \dot{x} = \frac{\delta \mathbf{v}^{\mathbf{T}} \mathbf{r_0} + \delta \mathbf{r}^{\mathbf{T}} \mathbf{v_0}}{r_0} - \frac{(\delta \mathbf{r}^{\mathbf{T}} \mathbf{r}_0)(\delta \mathbf{r_0^T} \mathbf{v}_0)}{r_0^3} \quad (14)$$

$$\delta \dot{y} = \frac{\delta \mathbf{v}^{\mathbf{T}} (\mathbf{H_0} \times \mathbf{r_0}) + \delta \mathbf{r}^{\mathbf{T}} (\dot{\mathbf{H}}_\mathbf{0} \times \mathbf{r_0} + \mathbf{H_0} \times \mathbf{v_0})}{|\mathbf{H_0} \times \mathbf{r_0}|} - \frac{\delta \mathbf{r}^{\mathbf{T}} (\mathbf{H_0} \times \mathbf{r_0})(\mathbf{H_0} \times \mathbf{r_0})^{\mathbf{T}} (\dot{\mathbf{H}}_\mathbf{0} \times \mathbf{r_0} + \mathbf{H_0} \times \mathbf{v_0})}{|\mathbf{H_0} \times \mathbf{r_0}|^3} \quad (15)$$

$$\delta \dot{z} = \frac{\delta \mathbf{v}^{\mathbf{T}} \mathbf{H_0} + \delta \mathbf{r}^{\mathbf{T}} \dot{\mathbf{H}}_0}{H_0} - \frac{\delta \mathbf{r}^{\mathbf{T}} \mathbf{H_0} (\mathbf{H_0^T} \dot{\mathbf{H}}_\mathbf{0})}{H_0^3} \quad (16)$$

We refer to relative motion along δx, δy, and δz, respectively, as radial, along-track, and out-of-plane.

Establishment of Approximate Periodic Motion Using Analytical Solutions

Schaub and Alfriend[11] have presented analytical results to establish orbital parameters of the deputy for a J_2 invariant relative orbit. It can be shown that for such orbits, the difference in the energies between the deputy and chief, to first order in J_2 is

$$E_1 - E_0 = \delta E = \frac{\mu J_2}{5} \frac{\delta H_z}{a_0 H_{z_0}^5} \cos^4 i_0 (1 + 5\cos^2 i_0) \quad (17)$$

and the difference in the polar component of the angular momentum is

$$\delta H_z = -\frac{5}{4} H_{z_0} \tan i_0 \delta i \quad (18)$$

where, the inclination difference between the two satellites is given by:

$$\delta i = i_1 - i_0 \quad (19)$$

Since δH_z and δE are known to be constants of motion in the absence of other perturbations, and can also be calculated using position and velocity information, they provide a means of checking transformations between mean and osculating orbit elements. The initial condition problem can be stated as follows:

Determine the position and velocity vectors of the deputy, given the position and velocity vectors of the chief, to satisfy the following constraints:

$$\dot{\mathbf{\Omega}}_0 = \dot{\mathbf{\Omega}}_1 \quad (20)$$

$$\dot{\boldsymbol{\omega}}_0 + \dot{M}_0 = \dot{\boldsymbol{\omega}}_1 + \dot{M}_1 \quad (21)$$

Since δi is specified, there are six unknown parameters to be selected with three constraints. Hence additional constraints can be imposed to get a unique solution. The constraints of Eq. (17-18) are equivalent to those of Eqs. (20-21). The process of converting the mean elements to the respective osculating elements is treated in Ref. 11. The conversion of the osculating elements to position and velocity can be found in any textbook on Astrodynamics, such as the one by Battin[13]. The nonlinear constraint satisfaction problem is solved using the nonlinear programming code NPOPT which is a part of the SNOPT package[14]. Two examples are treated in this section:

Example-1:

$$\mathbf{r}_0(0) = \begin{bmatrix} 5883.7397 \\ 2274.2509 \\ 2524.0672 \end{bmatrix} \text{ km}$$

$$\mathbf{v}_0(0) = \begin{bmatrix} -3.9273 \\ 4.5469 \\ 5.0533 \end{bmatrix} \text{ km/sec}$$

The mean elements for the chief, calculated using Brouwer's theory are:

$$\text{a} = 7152.9917 \text{ km}, \ \text{e} = .0499, \ \text{i} = 0.8377,$$
$$\Omega = -4\text{E-}7, \ \omega = 0.5237, \ \text{M} = -1.0037\text{E-}004$$

Besides the constraints given by Eqs. (20-21), three additional constraints are imposed as given below:

$$\delta i = .002 \ , \delta\Omega = .005 \ , \delta\omega = .01 \ , \text{ and } \delta M = -.01$$

The solution for the initial position and velocity vector offsets, expressed in the inertial frame are

$$\delta\mathbf{r}(0) = [\ -1.320\ \ -0.1061\ \ -0.511]\ \text{km}$$

$$\delta\mathbf{v}(0) = [-0.108\text{E-}2\ \ 0.377\text{E-}3\ \ 0.115\text{E-}2]\ \text{km/sec}$$

Figure 1 shows the relative orbit and Fig.2 shows the relative displacement. These figures represent data for a period of 3 days.

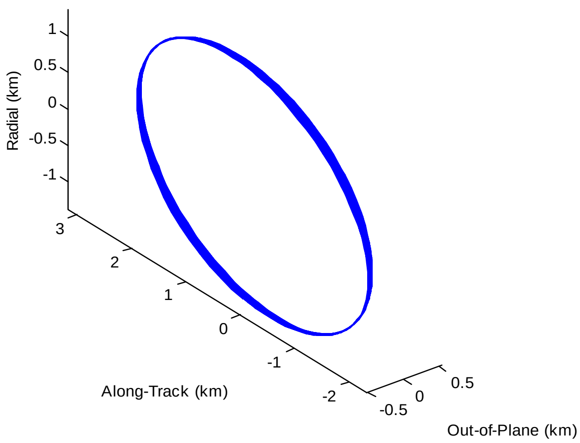

Fig. 1. Relative Orbits for Example-1

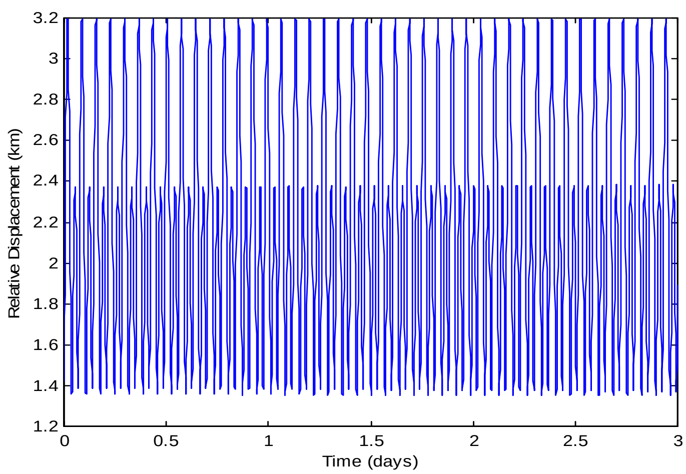

Fig. 2. Relative Displacement for Example-1

Example-2:

$$\mathbf{r}_0(0) = [7300\quad 0\quad 0]\,km$$

$$\mathbf{v}_0(0) = [0\quad 0.6\quad 8]\ \text{km/s}$$

This is an example of a near-polar orbit. Mean elements are:

$a_0 =$ 8876.786 km, $e_0 =$ 0.1781, and i_0 =1.49589. The geometric constraints are specified as

$$\delta i = \tan^{-1}(1/a_0), \delta\Omega = 0, \delta\omega = 0, \text{ and } \delta M = 0$$

The specification of the inclination difference as shown above, produces the maximum out-of-plane motion at the peak latitudes, in the amount of 1 km. The out-of-plane displacement at the equator is zero.

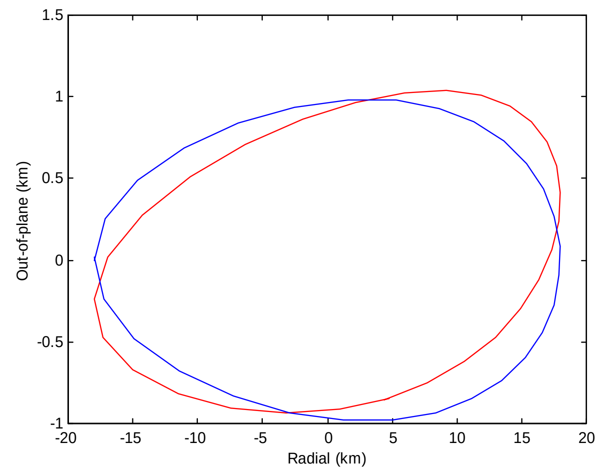

Fig. 3. Relative Orbits for Example-2

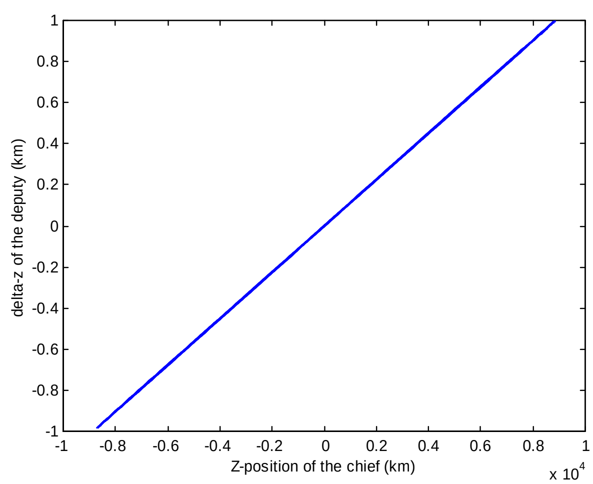

Fig. 4. Nature of the out-of-plane displacement for Example-2

The first and the last relative orbits are shown in Fig. 3 over a period of 10 days. Note that the axes are not drawn to the same scale, in order to exaggerate the out-of-plane motion. The relationship between δz , the out-of-plane displacement and z , the polar component of the chief's inertial displacement are plotted in Fig. 4. This figure shows that indeed, the maximum out-of-plane displacement of 1 km is achieved at the extreme latitudes.

Establishment of Periodic Motion by Numerical Integration

In this section the procedure to obtain periodic or near-periodic relative motion between the chief and the deputy is described. Given the initial conditions of the chief, initial conditions of the deputy are sought such that the relative velocity and the relative position vectors satisfy periodicity conditions at an unknown time (approximately the nodal period of the chief's orbit). Hence seven parameters have to be determined.

The six constraints on the relative positions and velocities are of the type given by Eqs. (22-23)

$$\delta y(0) = \delta y(t_f) \tag{22}$$

$$\delta \dot{x}(0) = \delta \dot{x}(t_f) \tag{23}$$

where t_f is an undetermined parameter. In order to compare the solutions to the same problem using the the current and the previous method, Example-2 is selected for testing. Notice that in Fig. 4, there is a linear relationship between δz and z. This requires two constraints as shown below:

$$\tan(\delta i)\, z_0 - \sin(i_0)\delta z = 0 \tag{24}$$
$$\tan(\delta i)\, \dot{z}_0 - \sin(i_0)\delta \dot{z} = 0 \tag{25}$$

These constraints can be specified at the initial or final times. Since there is one too many constraints, the radial position periodic constraint is dropped. This gives rise to a problem with seven unknowns and seven constraints. The initial guess for the solution is obtained from the solution to the same problem using the method described in the previous section.

The relative orbits for Example-2 for a period of 10 days are shown in Fig. 5. Only the first and the last orbits are shown in the figure for the sake of clarity. Figure 6, shows the plot of the out-of-plane displacement versus the polar component of the displacement of the chief. It is interesting to note that the relative orbits of Fig. 5 are larger compared to those of Fig. 3 but the drift is much less. However, Fig. 6 shows that there is a slight differential nodal precession. This is due to the way the constraints are specified. In the first method, equal nodal rates are enforced. Nodal precession is about the inertial z-axis. In the second method, the constraints are in terms of the rotating frame, in which Figs. 1, 3, and 5 are plotted. Although the integration method is more accurate, it is inconvenient for computing optimal controls.

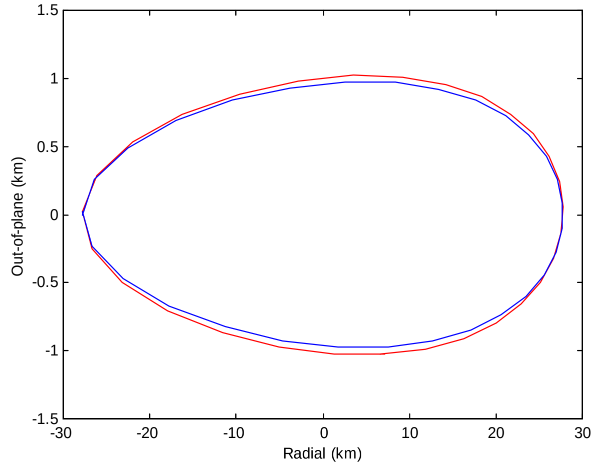

Fig. 5. Relative Orbits for Example-2 obtained using the integration method

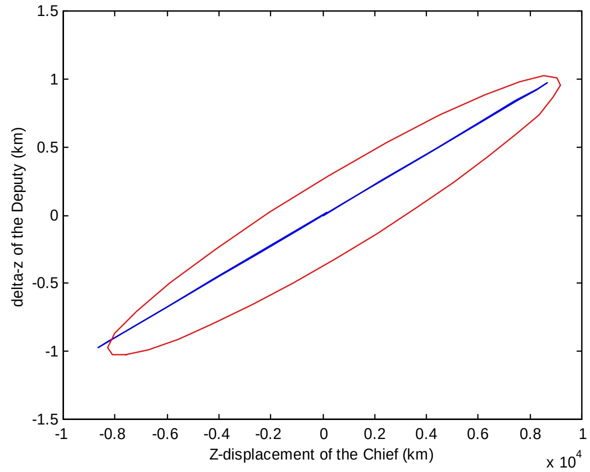

Fig. 6. Nature of the out-of-plane dispalcement for Example-2

Fuel-Optimal Multi-Impulse Control

In this section, the first method of determining initial conditions, discussed above, is used to define a terminal constraint set for the multi-impulse fuel-optimal control problem. Prussing and Chiu15 have utilized primer vector theory for computing optimal multi-impulse rendezvous maneuvers. Kumar and Seywald[16] treat the problem of fuel-optimal station-keeping based on Hill's equations. Smith[17] has studied the problem of satellite constellation maintenance in a multi-impulse setting using genetic algorithms. Schaub et al.[18] present a feedback control scheme using the mean elements as well as the cartesian position and velocity vectors. Ulybyshev[19] uses the LQR approach for controlling the drifts in period and nodes. A nonlinear, adaptive, tracking controller is proposed by Queiroz et al[20]. In this paper, we use the mean element formulation and Gauss' variational equations[13]. These equations are quite convenient for the problem at hand due to the fact that during coasts, the mean a, e, and i remain constant and the mean $\boldsymbol{\Omega}, \boldsymbol{\omega}$, and M vary linearly. Gauss's equations for the variations in the mean elements are written below:

$$\dot{\mathbf{e}} = \mathbf{A}(\mathbf{e}) + \mathbf{B}(\mathbf{e})\mathbf{u} \tag{26}$$

where, $\mathbf{e} = [a\ e\ i\ \Omega\ \omega M]^T$ and the control $\mathbf{u} = [u_r\ u_\theta u_\mathrm{H}\ ]^T$. The control vector is defined using components along the radial, tangential, and orbit normal directions. Other quantities are as defined below:

$$\mathbf{A}(\mathbf{e}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1.5 J_2 \left(\dfrac{R_e}{p}\right)^2 n \cos i \\ 0.75 J_2 \left(\dfrac{R_e}{p}\right)^2 n\left(5\cos^2 i - 1\right) \\ n + 0.75 J_2 \sqrt{1-e^2}\left(\dfrac{R_e}{p}\right)^2 n\left(3\cos^2 i - 1\right) \end{bmatrix}$$

The above matrix is obtained by using the average drift rates for the elements. The $\mathbf{B}(\mathbf{e})$ matrix is written in the same form as given in Ref. 13, with the assumption that mean orbit elements will be used to valuate it. This may not be precise but proves useful.

$$\mathbf{B}(\mathbf{e}) = \begin{bmatrix} \dfrac{2a^2 e \sin f}{H} & \dfrac{2a^2 p}{Hr} & 0 \\ \dfrac{p \sin f}{H} & \dfrac{(p+r)\cos f + re}{H} & 0 \\ 0 & 0 & \dfrac{r\cos\theta}{H} \\ 0 & 0 & \dfrac{r\sin\theta}{H\sin i} \\ \dfrac{-p\cos f}{He} & \dfrac{(p+r)\sin f}{He} & \dfrac{-r\sin\theta\cos i}{H\sin i} \\ \dfrac{\eta(p\cos f - 2re)}{He} & \dfrac{-\eta(p+r)\sin f}{He} & 0 \end{bmatrix}$$

$$\boldsymbol{\eta} = \sqrt{1-e^2},\ r = \frac{p}{1+e\cos f},\ \text{and } \boldsymbol{\theta} = \boldsymbol{\omega} + f$$

$H = \sqrt{\mu p}$ is the scalar angular momentum. The true anomaly, f, is related to the mean anomaly through the following equations:

$$M = E - e\sin E \tag{27}$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{28}$$

In order to establish a formation, the mean elements a, e, and i need satisfy Eqs. (19-21) and the mean $\boldsymbol{\Omega}, \boldsymbol{\omega}$, and M have to meet some specifications. Assuming that the control is impulsive, the elements undergo jump discontinuities at the impulse application times. During the coasting phases, they can be integrated analytically. The true anomaly and the other time-varying quantities in the $\mathbf{B}(\mathbf{e})$ matrix can be calculated using the equations presented above. The impulse application times and the delta-v magnitude/directions become the free parameters. Since the fuel consumed is directly proportional to delata-v, the net delta-v is to be minimized. Once the optimization process is over, the terminal position and velocity states of the deputy can be obtained by transformation from the mean elements. The maximum number of impulses is unknown apriori, but is not more than six since there are six elements to be changed in general. Allowing for initial and final coasts and six impulses, the total number of parameters is 25. In some instances, a computed impulse magnitude may be too small and can be eliminated.

Example:

The mean elements of the chief are selected as

$$\mathbf{e_0} = [7555\text{km} \ .05 \ 48^0 \ \ 0 \ 10^0 \ \ 120^0]$$

The corresponding orbital elements of the deputy are ,

$$\mathbf{e}_1 = [7555\text{km} \ .05057 \ 48.054^0 \\ -0.01 \ \ 9.9981 \ \ 119.9774 \ ]$$

The desired orbit element differences at the time of orbit establishment are

$$\delta i = .006^0, \ \ \delta\Omega = \delta\omega = \delta M = 0^0$$

The differences in the other orbit elements to satisfy the two rate constraints can be determined from Eq. (20-21)as

$$\delta a = -.00193km, \ \ \delta e = .0005767$$

The deputy is initially disturbed from the desired state by introducing large errors in $a, i,$ and Ω :

$$a_1 = a_0 + \delta a - 100m$$
$$i_1 = i_0 + \delta i + .05^0$$
$$\Omega_1 = \Omega_0 + \delta\Omega - .01^0$$
$$\mathrm{e}_1 \ = \mathrm{e}_0, \omega_1 \ = \omega_0, M_1 = M_0$$

A six-impulse solution to the orbit establishment problem is given below in Table 1:

Table1

Impulse Magnitudes

#	Radial	Tangential	Normal	Total (km/sec)
1	0.51D-4	0.40D-4	0.278D-2	2.78D-3
2	-0.57D-7	0.442D-7	0.758D-6	7.62D-7
3	0.153D-6	-0.125D-4	-0.26D-4	2.92D-5
4	-0.11D-7	0.918D-8	-0.74D-8	1.65D-8
5	0.55D-4	-0.184D-4	0.338D-2	3.38D-3
6	-0.42D-6	-0.51D-5	-0.98D-5	1.11D-5

The total delta-v required is 6.204 m/sec, which is primarily due to impulses 1 and 5. Both of them are predominantly in the normal direction. They occur when θ is close to π. The $\mathbf{B}(\mathbf{e})$ matrix shows that this is a time when inclination change is best performed.

The impulse times and the final time are

$$t_i = [640.49, \ \ 813.467, \ 3863.48 \ , \\ 6014.13, \ 7169.23, 1047.85]\text{sec}$$
$$t_f \ = 1.19602\text{d4 sec}$$

The variation of the semi-major axis is shown in Fig. 7. The first impulse overcorrects for the error and the subsequent impulses bring the semi-major axis back to the desired value.

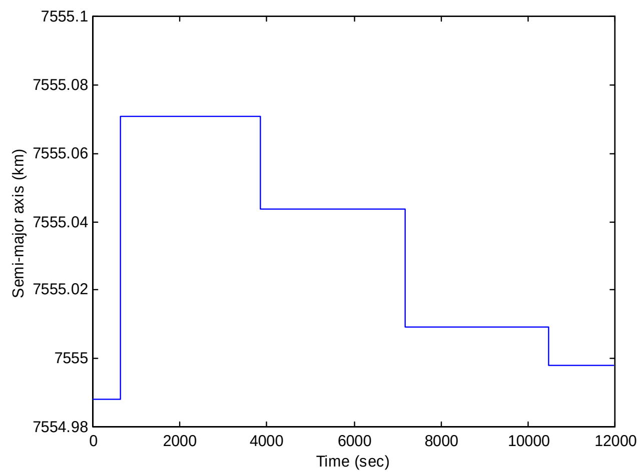

Fig. 7. Changes in the Semi-major axis

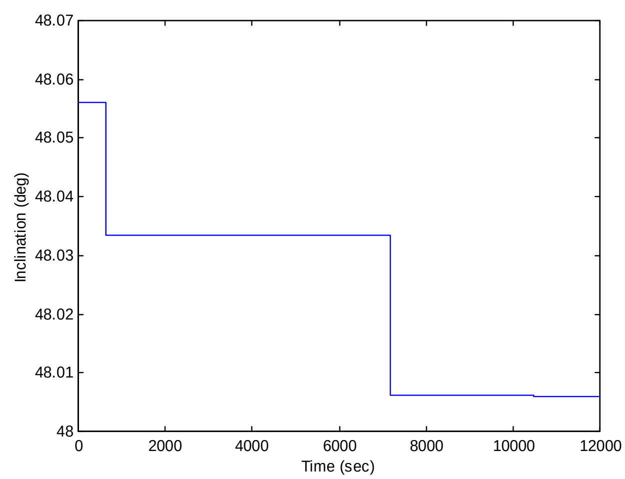

Fig. 8. Changes in the inclination

Figure 8 shows that the changes in the inclination come in two installments. As mentioned before, the delta-v expenditure is primarily due to these inclination changes. Figure 8 shows the resulting relative orbits over a period of 3 days after the maneuver is completed.

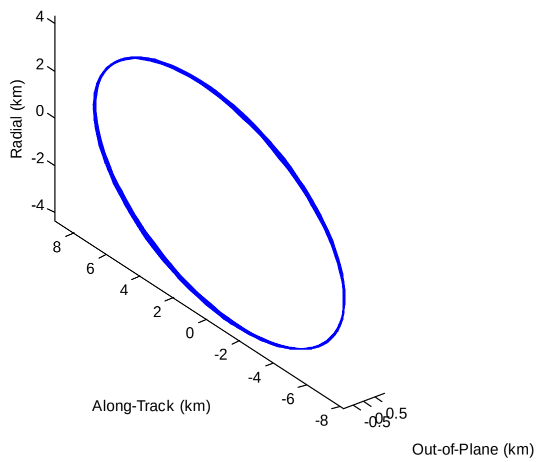

Fig. 9. Relative Orbits after the maneuver

Orbit Maintenance Using Power-Limited, Electric Propulsion

Electric propulsion using ion or plasma thrusters is being considered for formation flying. Power- limited, low-thrust propulsion has already been demonstrated by the DS-1 mission. The Techsat-21[23] program has a proposed mission plan, involving swarms of microsatellites, each with an estimated mass of 77 kg and a power rating of 1 kw. The thruster rating for each satellite is of the order of 0.1 N. The specific impulse, Isp, can vary between 1200-2500 sec. It is assumed here that each satellite has one thruster that can be gimbaled to produce thrust in a desired direction.

There exist many works in this area. Kechichian[21] has presented an excellent treatment of orbit transfer using low-thrust propulsion. Coverstone-Carroll and Prussing[22] consider the problem of cooperative power-limited rendezvous between two satellites in neighboring circular orbits using Hill's framework. In this study both the satellites are given the freedom to maneuver and cooperate to achieve better performance than that of an active/passive pair.

Since the available thrust is severely limited for the Techsat 21 program, the Isp bounds must be considered. The formulation presented here is similar to that given in Ref. 21. Only the deputy is assumed to be active.

The equations of motion including the thrust terms are given below:

$$\dot{\mathbf{r}} = \mathbf{v} \tag{29}$$

$$\dot{\mathbf{v}} = -\phi_r + \frac{2\boldsymbol{\varepsilon} P}{mg}\mathbf{u} \tag{30}$$

$$\dot{m} = -\frac{2\boldsymbol{\varepsilon} P}{g^2}\mathbf{u}^T\mathbf{u} \tag{31}$$

where P, is the maximum power available, $\boldsymbol{\varepsilon}$, is the efficiency factor, g, is the acceleration due to gravity at sea level, m, is the mass of the satellite, and $\mathbf{u}$ is the control vector defined as follows:

$$\mathbf{u}^T\mathbf{u} = \frac{1}{Isp^2}$$

The controls are constrained through the relationship given below:

$$\frac{1}{Isp_{\max}{}^2} \le \frac{1}{Isp^2} \le \frac{1}{Isp_{\min}{}^2} \tag{32}$$

The optimal control problem is posed as:

$$\text{Minimize: } J = -m(t_f) \tag{33}$$

subject to Eqs. (29-31) and the terminal constraints $\boldsymbol{\Psi}(\mathbf{e}(t_f), t_f) = 0$, where, $\mathbf{e}(t_f)$, is the mean orbit element specification for the deputy. The constraint set is of dimension six and is full rank. It is the same used in method-1 for the determination of initial conditions. The final time, t_f, is free. The variational Hamiltonian is written as follows:

$$\mathrm{H} = \boldsymbol{\lambda}_{\mathbf{r}}\dot{\mathbf{r}} + \boldsymbol{\lambda}_{\mathbf{v}}\dot{\mathbf{v}} + +\lambda_m \dot{m} \tag{34}$$

The costate equations are

$$\boldsymbol{\lambda}_{\mathbf{r}} = -\frac{\partial \mathrm{H}}{\partial \mathbf{r}} \tag{35}$$

$$\boldsymbol{\lambda}_{\mathbf{v}} = -\frac{\partial \mathrm{H}}{\partial \mathbf{v}} \tag{36}$$

The controls are related to the costates as shown below[24]:

Unconstrained Solution:

$$\mathbf{u} = \frac{g}{2m}\frac{\boldsymbol{\lambda}_{\mathbf{v}}}{\lambda_m} \tag{37}$$

The convexity condition is

$$\frac{\partial^2 H}{\partial \mathbf{u}^2} = -\frac{4\varepsilon P}{g^2}\lambda_m > 0 \tag{38}$$

Hence the solution as obtained above is optimal if $\lambda_m < 0$ and the Isp constraint is not violated. For a minimum-fuel problem, λ_m is negative if the final time is sufficiently long. However for the problem at hand, this may not be the case. The optimal control when the constraints are active are given below:

Constrained Solutions

For $\lambda_m > 0$, the optimal solution requires minimum Isp. This solution is

$$\mathbf{u} = -\frac{\boldsymbol{\lambda}_{\mathbf{v}}}{I_{sp_{\min}}\sqrt{\boldsymbol{\lambda}_{\mathbf{v}}^T\boldsymbol{\lambda}_{\mathbf{v}}}} \tag{39}$$

For $\lambda_m < 0$, there are two possibilities. If the unconstrained solution is such that $I_{sp} > I_{sp_{\max}}$ then the controls are given by

$$\mathbf{u} = -\frac{\boldsymbol{\lambda}_{\mathbf{v}}}{I_{sp_{\max}}\sqrt{\boldsymbol{\lambda}_{\mathbf{v}}^T\boldsymbol{\lambda}_{\mathbf{v}}}} \tag{40}$$

If the unconstrained solution is such that $I_{sp} < I_{sp_{\min}}$ then the controls are the same as in Eq. (39).

The transversality condition on the mass costate is

$$\lambda_m(t_f) = -1 \tag{41}$$

Since the transformation between the position and velocity vectors and the mean elements is quite complex, the free-final time, terminally constrained, optimal control problem is solved by a direct approach using the nonlinear programming code NPOPT. The transversality conditions on the other costates and the Hamiltonian are not utilized. There are seven unknown initial costates and the final time, to be determined. The nonlinear programming problem is posed with the performance index given by Eq. (33). The performance index and the terminal constraints are evaluated by integrating Eqs. (29-31 and 35-36). The controls are evaluated using Eqs. (37), (39), or (40), which ever is applicable. The terminal constraints are given by Eqs. (19-21), three specifications on $\boldsymbol{\Omega}, \boldsymbol{\omega}$, and M, and Eq. (41). The partial derivatives are computed using finite differences by the code.

As an example, the initial conditions of the chief are chosen as those of Example-1 on page-3. The desired mean orbit element differences are:

$$\delta i = .00057,\ \delta\Omega = .01,\ \delta\omega = .01,\ \text{and}\ \delta M = -.01$$

The initial conditions of the deputy are disturbed from the desired, by perturbing its position by 100 m along each inertial axis and the inertial velocity by 0.1 m/sec along each axis. To correct for the initial errors, the minimum Isp of 1200 sec is required. The final time is 399.3 sec and the mass of fuel required is 0.0029 kg. The control accelerations required along the three inertial directions are those shown in Fig. 10.

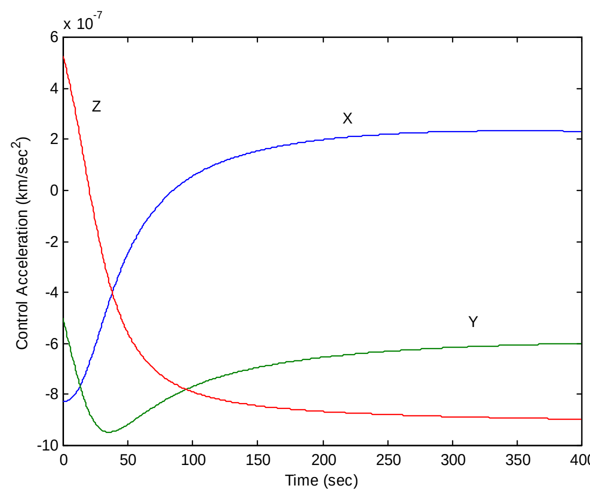

Fig. 10. Controls along the inertial axes

The maximum control acceleration required is of the order of $0.0011\,\mathrm{m/sec}^2$. This translates to a peak thrust magnitude of .085 N at beginning of life of the satellite.

Conclusions

In this paper the formulation and solution of the initial condition determination and fuel-optimal control problem in regard to formation flying of satellites was presented. The emphasis was on creation of the out-of-plane relative motion using both inclination and node differences. Two methods for determining initial conditions were presented. The first method matches the mean J_2-induced angular drift rates of the two satellites, and the second method is based on imposing periodic boundary conditions on the relative position and velocity in a rotating coordinate system. Even though the second method is more accurate, the first method can readily be used for control. A method to determine multi-impulse fuel-optimal impulsive maneuvers to establish formations was presented. A fuel-optimal, low-thrust, variable Isp propulsion scheme was presented for orbit maintenance.

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