AAS 19-684

ANALYTIC APPROXIMATIONS OF ORBIT GEOMETRY IN A ROTATING HIGHER ORDER GRAVITY FIELD

Ethan R. Burnett* **and Hanspeter Schaub**†

This paper introduces new analytic approximations of the orbital state for a subset of orbits in a rotating potential with gravitational harmonics $C_{20} = -J_2$ and C_{22}. An analytic expression for the orbit radius is first obtained, then used to obtain expressions for 3 other quantities, which may be combined with equations for the right ascension of the ascending node and inclination to fully characterize the orbital state. The approximations are fully developed for near-circular orbits with initial mean motion n_0 around a body with rotation rate c. The approximations are shown to be valid for values of $\Gamma = c/n_0 > 1$, with accuracy decreasing as $\Gamma \to 1$, and singularities at $\Gamma = 1$. The methodology in this paper can be adapted to approximate eccentric orbits in more general asymmetric potentials, and the necessary modifications are discussed.

INTRODUCTION

Orbital motion in uniformly rotating irregular gravity fields is generally non-integrable, greatly complicating the task of characterizing and studying the system behavior without relying on numerical simulation. Because orbits about asteroids and other small bodies can often be well-understood by considering the effects of C_{20} and C_{22} (along with solar radiation pressure and third-body effects in some situations),[15] studying motion in this particular truncated gravitational potential is a topic of significant interest for astrodynamicists and other scientists who research such bodies. Analytical developments in this problem lend useful insight for the study of orbital mechanics in the complex gravity fields of asteroids and other small bodies, which are growing targets for scientific exploration.

By ignoring the effect of sectoral harmonics such as C_{22} (suitable in some situations for orbits about planets), the time-varying aspect of the gravitational potential is removed. This sub-problem is far simpler and more relevant to planetary orbiters and Earth satellites. Thus, there has been a lot of work in analyzing orbital motion in the axisymmetric potential of an oblate planet. Particularly noteworthy are the influential works of Brouwer[4] and Kozai,[10] as well as Vinti,[17] who approximates the effects of the axisymmetric gravity field using his intermediary potential. Some more recent work is also relevant to this discussion, including work by Martinusi et al.[11,12] using an averaging technique and Brouwer-Lyddane theory to approximate the motion of low-Earth orbiting satellites, and exploiting the superintegrability of equatorial orbital dynamics under the influence of even zonal harmonics to obtain analytic expressions in terms of elliptic integrals.

*Research Assistant, Ann and H.J. Smead Department of Aerospace Engineering Sciences, University of Colorado Boulder, Boulder, CO, 80309 USA.
†Glenn L. Murphy Chair of Engineering, Smead Department of Aerospace Engineering Sciences, University of Colorado, 431 UCB, Colorado Center for Astrodynamics Research, Boulder, CO 80309-0431. AAS Fellow, AIAA Fellow.

Plenty of analysis has been dedicated to the problem of motion in a second degree and order gravity field in the primary body-fixed rotating frame.[7] Work has also been done to analyze this complex problem from the perspective of the perturbed osculating orbit in a non-rotating frame,[15,16] and numerical studies have investigated the problem in the inertial frame.[6] However, explicit analytic approximations of the orbit radius have not yet been obtained. In this paper, the variations in the orbit radius in a rotating second degree and order gravity field are described for a subset of orbit cases in a manner analytic in time or an angle. The kinematics of the osculating orbit are used to obtain an approximate scalar differential equation for the orbit radius r, which is rendered as a time-varying differential equation in r alone to first order in small variations. This is done using the existence of conserved quantities. In the case of the C_{20}-only potential, the orbit energy is used, while the Jacobi integral is used for the case of the rotating potential with nonzero C_{20} and C_{22}. The resulting approximate solutions are shown to be highly accurate in the applicable parameter space, while also enabling additional analytical descriptions of variations in other elements, to fully characterize the orbital state.

Combining the results of the orbit radius and argument of latitude θ (and their rates) with the analytic approximations of the right ascension of the ascending node Ω and inclination i, the orbital state can be fully described. This paper explores how to consider the first-order effects of the rotating gravity field to obtain analytic descriptions that are explicit functions of initial conditions and either time or an angle, and the approach can be adapted to higher order in a perturbative approach. This paper presents an approximate solution of the orbital state for the problem of near-circular orbits in the rotating potential, and also includes a discussion of refinements of the analytical approach to treat eccentric orbits as well.

The approximate solutions in this paper could find use in mission design and in astrodynamics research. In particular, analytic approximations of orbit behavior in rotating higher-order gravity fields will provide an additional tool of analysis for a problem that typically forces the astrodynamicist to rely almost exclusively on numerical simulation. The form of terms in the approximate solution should also provide some insight into the dynamics of orbits about quickly rotating asteroids.

ORBITS IN A ROTATING GRAVITY FIELD

In this section, the dynamical problem of interest is introduced: orbital motion about a uniformly rotating body with nonzero ellipticity captured by the C_{22} coefficient. The problem geometry is given, followed by discussion of the analytical challenges of obtaining analytic approximations of orbital motion in this potential.

Problem Geometry

Relevant geometry for this problem is reproduced in Figure 1. The main orbit equation of interest for this work is the motion of a spacecraft in orbit in a rotating primary body-fixed second degree and order gravity field:[15]

$$\ddot{\boldsymbol{r}} = \boldsymbol{a}_{C_{00}} + \boldsymbol{a}_{C_{20}} + \boldsymbol{a}_{C_{22}} \tag{1}$$

$$\boldsymbol{a}_{C_{00}} = -\mu \frac{\boldsymbol{r}}{r^3} \tag{2}$$

$$\boldsymbol{a}_{C_{20}} = \frac{3\mu C_{20} R^2}{2r^4} \left[\left(1 - 5\left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_3\right)^2\right) \hat{\boldsymbol{e}}_r + 2\left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_3\right) \hat{\boldsymbol{a}}_3 \right] \tag{3}$$

$$\boldsymbol{a}_{C_{22}} = \frac{3\mu C_{22} R^2}{r^4} \left[-5\left(\left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_1\right)^2 - \left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_2\right)^2\right) \hat{\boldsymbol{e}}_r + 2\left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_1\right) \hat{\boldsymbol{a}}_1 - 2\left(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{a}}_2\right) \hat{\boldsymbol{a}}_2 \right] \tag{4}$$

where $\boldsymbol{r}$ is the radial vector to the orbiting satellite and $\hat{\boldsymbol{a}}_1$ and $\hat{\boldsymbol{a}}_2$ are aligned with the minimum and

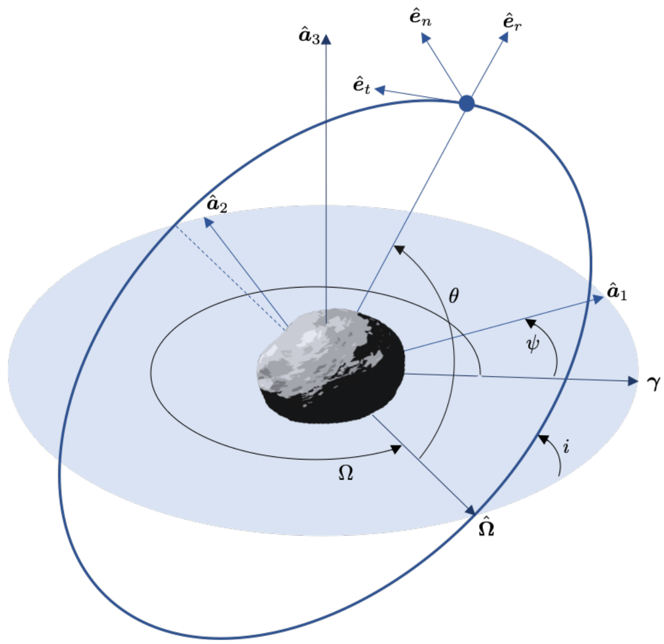

Figure 1. Problem Geometry[1]

intermediate principal axes of inertia for a body in a stable spin state. This analysis does not consider more complicated spin states. C_{ij} are coefficients from a spherical harmonics series expansion of the gravitational potential, and $C_{20} = -J_2$ is due to the oblateness of the body while C_{22} is due to the ellipticity. These two second degree and order coefficients are typically on the order of $\mathcal{O}(10^{-3})$ for planets and up to $\mathcal{O}(10^{-2})$ for asteroids. For asteroids, they are typically the dominant secondary values in the gravitational potential after the bulk mass $C_{00} \equiv 1$ contribution.

Instead of describing the state of the orbit using the orbit position $\boldsymbol{r}$ and velocity $\dot{\boldsymbol{r}}$, the orbit can be described with the set of classical orbit elements, $a, e, i, \omega, \Omega, f$ (semimajor axis, eccentricity, inclination, argument of periapsis, right ascension of the ascending node, true anomaly), where $\theta = \omega + f$ is the argument of latitude. For unperturbed circular orbits, this orbit angle varies with the mean motion $n = \sqrt{\mu/a^3}$. Of course, under the presence of orbit perturbations, all of the orbit elements are generally time-varying. Note that the angle ψ is the rotation angle of the primary body, which rotates with a constant velocity c about an axis that points in a fixed direction in space.

Analytic Challenges

This paper is focused on two major challenges of approximating the orbital behavior in the given situation. The first challenge is the general increase in complexity from the zonal problem introduced by the rotating gravity field. The time-varying nature of the potential results in the classical orbit energy term $E = \frac{1}{2}v^2 - U(r)$ not being conserved. Only the more complex Jacobi integral is conserved. Furthermore, the disparate behaviors of the perturbed orbit and the uniform rotation of the body also lead to analytical difficulty. Namely, the problem dynamics are determined by the orbit plane configuration, the primary body orientation angle ψ (which is linear in time in this analysis), and by an orbit latitude or anomaly which is generally quite nonlinear in time for eccentric orbits, and is itself affected by perturbations of the rotating body. To analytically describe solution behavior in terms of initial conditions and a single independent variable, the different angular behaviors must be reconciled.

For cases with a consistently near-circular orbit, additional challenges are introduced. In such cases, the behavior of the argument of latitude is approximately linear in time, $\theta \approx \theta_0 + nt$, but the classical orbit description in terms of an osculating ellipse becomes less useful. In particular, the classical elements of eccentricity, argument of periapsis, and true anomaly oscillate rapidly in a manner that cannot be approximated by considering small variations about a mean or initial value. The eccentricity also appears as a small divisor in the formulas for the argument of periapsis and true or mean anomalies. These difficulties can be removed by considering alternate element formulations in terms of the troublesome classical elements, such as the equinoctial elements[3] or Poincaré canonical elements. By the same manner, these alternate elements are not subject to large and rapid variations due to perturbations, and are thus potentially more suitable as a choice of coordinates in these situations than the classical elements. However, the variational equations for the equinoctial elements must typically be numerically integrated using Kepler's equation at each integration step,[3] an unfortunate property of the independent variable for the purposes of this work. For the perturbed near-circular orbit problem, one would expect more mathematical simplicity instead of complexity.[8] A concise analytic approximation demands a simpler choice of coordinates. The approach in this paper uses the classical elements Ω and i to describe the orientation of the perturbed orbit plane, and polar coordinates (r, θ, $\dot{r}$, ω_n) to parameterize the remaining state elements. This treatment avoids any direct use of the eccentricity, argument of periapsis, or true anomaly, and any associated difficulties of using elements explicitly derived from these. It also remains well-defined for equatorial orbits simply by redefining θ as the rotation from $\hat{\boldsymbol{\gamma}}$ instead of the undefined ascending node. While particularly well-suited for the small eccentricity problem, this approach can also be used with more eccentric orbits.

The first contribution in this paper is to obtain a scalar nonlinear differential equation for the orbit radius that is explicit in the argument of latitude, θ, and body orientation angle, ψ, using assumptions about orbit plane kinematics and conservation of an integral. This equation is a good approximation for bound orbits with reasonably small eccentricity (e.g. $e < 0.3$). It can be solved directly in a perturbative manner without directly considering the variations in any other elements. The analytic orbit description in this paper is obtained using a first-order perturbative approach to study small deviations from the initial radius r_0 for near-circular orbits. The extension to higher-order solutions is discussed, along with other approximation approaches.

EXPRESSIONS FOR THE PERTURBED ORBIT RADIUS

It is possible to isolate the behavior of the orbit radius by balancing the centripetal acceleration with the radial component of the perturbation accelerations. This begins by equating the acceleration in local radial (along $\hat{\boldsymbol{e}}_r$), transverse ($\hat{\boldsymbol{e}}_t$), and normal ($\hat{\boldsymbol{e}}_n$) components with the gravitational acceleration terms, noting $(\;)' = \frac{^{\mathcal{H}}\mathrm{d}}{\mathrm{d}t}$ (time-derivative in the orbiter-fixed rotating $\mathcal{H}$ frame), and $\boldsymbol{\omega}_O$ is the instantaneous angular velocity of the orbiter:

$$\boldsymbol{r}'' + 2\boldsymbol{\omega}_O \times \boldsymbol{r}' + \boldsymbol{\omega}_O' \times \boldsymbol{r} + \boldsymbol{\omega}_O \times (\boldsymbol{\omega}_O \times \boldsymbol{r}) = \boldsymbol{a}_{C_{00}} + \boldsymbol{a}_{C_{20}} + \boldsymbol{a}_{C_{22}} \tag{5}$$

The instantaneous angular velocity is given below,[5,9] followed by the analytical rates of the perturbed Ω and i, obtained using Gauss' form of the variational equations:[1]

$$\boldsymbol{\omega}_O = (\dot{\Omega}\sin i \sin\theta + \dot{i}\cos\theta)\hat{\boldsymbol{e}}_r + \left(\dot{\theta} + \dot{\Omega}\cos i\right)\hat{\boldsymbol{e}}_n \tag{6}$$

$$\dot{\Omega} = \frac{3\mu R^2}{hr^3}\left(C_{22}\sin\left(2(\Omega-\psi)\right)\sin 2\theta + \left[C_{20} + 2C_{22}\cos\left(2(\Omega-\psi)\right)\right]\cos i\sin^2\theta\right) \tag{7}$$

$$\dot{i} = \frac{3\mu R^2}{hr^3}\left(2C_{22}\sin\left(2(\Omega-\psi)\right)\cos^2\theta\sin i + \frac{1}{4}\left[C_{20} + 2C_{22}\cos\left(2(\Omega-\psi)\right)\right]\sin 2\theta\sin 2i\right) \tag{8}$$

In this paper, let ω_n denote the "angular rate" around the orbit, $\omega_n = \boldsymbol{\omega}_O \cdot \hat{\boldsymbol{e}}_n$. This is related to the argument of latitude rate by $\dot{\theta} = \omega_n - \dot{\Omega}\cos i$, where the second term is due to the deviation and regression of the node from which θ is measured.[13] The angular rate ω_n is one of the six chosen state quantities, while the argument of latitude rate $\dot{\theta}$ is not. Note that the velocity of the orbiter in the orbit plane is defined as $v = \sqrt{\dot{r}^2 + r^2\omega_n^2}$, because the $\hat{\boldsymbol{e}}_r$ component of $\boldsymbol{\omega}_O$ does not contribute to the orbital velocity.

The angular rates $\dot{\Omega}$ and $\dot{i}$ are of the following scale or smaller, assuming $|C_{20}| \geq |C_{22}|$:

$$\dot{\Omega}, \dot{i} = \mathcal{O}\left(C_{20}\left(\frac{R}{r}\right)^2\left(\frac{1}{\rho^3}\right)\frac{n^2}{\omega_n}\right) \tag{9}$$

where $\rho = r/a$, $\omega_n = \mathcal{O}(n)$, and it is assumed $|C_{20}(R/r)^2| \ll 1$. In orbits for which this assumption holds, if the eccentricity is reasonably small, then $|\omega_n| \gg |\dot{\Omega}|, |\dot{i}|$.

Substituting Eq. (6) into Eq. (5) and dotting the vector equation by $\hat{\boldsymbol{e}}_r$, the following is obtained:

$$\ddot{r} - \left(\dot{\theta}^2 + 2\dot{\theta}\dot{\Omega}\cos i + \dot{\Omega}^2\cos^2 i\right) r = -\frac{\mu}{r^2} + R_{C_{20}} + R_{C_{22}} \tag{10}$$

where $R_{C_{ij}} = \boldsymbol{a}_{C_{ij}} \cdot \hat{\boldsymbol{e}}_r$, and all terms containing the angular rates are quadratic in these terms. No approximations of the original problem are made to obtain Eq. (10).

Substituting $\dot{\theta} = \omega_n - \dot{\Omega}\cos i$, Eq. (10) simplifies to the following:

$$\ddot{r} - \omega_n^2 r = -\frac{\mu}{r^2} + R_{C_{20}} + R_{C_{22}} \tag{11}$$

The following time-varying differential equation is obtained for the orbit radius by substituting the radial components of the disturbance accelerations:

$$\begin{aligned}\ddot{r} - \omega_n^2 r = &-\frac{\mu}{r^2} + \frac{\mu}{r^4}\left(\frac{3}{2}C_{20}R^2\left(1 - 3\sin^2 i\sin^2\theta\right) + 3C_{22}R^2\left(3\sin\left(2\left(\Omega-\psi\right)\right)\cos i\sin 2\theta\right.\right.\\ &\left.\left.-\frac{3}{4}\cos\left(2\left(\Omega-\psi\right)\right)\left(1 + 3\cos 2\theta - 2\cos 2i\sin^2\theta\right)\right)\right)\end{aligned} \tag{12}$$

To isolate the dynamics of the orbit radius, the currently unknown ω_n term in Eq. (12) must be rewritten. This is done by isolating the ω_n terms in an integral of motion and re-arranging to obtain an expression for ω_n that is a function of only r, $\dot{r}$, θ, Ω, i, ψ, and the conserved value of that integral of motion. In the expression for ω_n, functions of quantities Ω and i primarily appear pre-multiplied by small parameters that are functions of C_{20} and C_{22}. Then, the assumption that $\dot{\Omega}$ and $\dot{i}$ are "small" equivalently results in using the initial values Ω_0 and i_0 in these terms. In the following sections, this equation will be solved using conservation of energy for the case $C_{22} = 0$ for all inclinations, and it will be solved for an expression accurate for inclinations below a critical value for $C_{22} \neq 0$ using conservation of the Jacobi integral.

Approximate Solution Using Conservation of Energy for Near-Circular Orbits

To introduce the perturbative procedure for approximating the orbit radius, the simpler C_{20}-only zonal problem is first solved. For the case of negligible influence by C_{22}, using the orbit energy and the substitutions $r(t) \approx r_0(1+\xi(t))$ where $\xi \sim \mathcal{O}(1/r_0)$ and $\dot{r} \approx r_0\dot{\xi}$, it is possible to approximate the behavior of Eq. (12) with a simpler differential equation that is an explicit function of time. Below, the total orbit energy (only conserved when either C_{22} or c are zero) is given, where $U(r)$ is the gravitational potential:

$$E = \frac{1}{2}v^2 - U(r) \tag{13}$$

$$\begin{aligned} U(r) = U(r,\theta,\psi) = \frac{\mu}{r} + \frac{\mu}{r^3}\Bigg[& C_{20}R^2\left(\frac{3}{4}\sin^2 i\,(1-\cos 2\theta) - \frac{1}{2}\right) \\ & + 3C_{22}R^2\left(\frac{1}{2}\sin^2 i\cos\left(2(\Omega-\psi)\right) + \cos^4\left(\frac{i}{2}\right)\cos\left(2(\Omega+\theta-\psi)\right)\right. \\ & \left. + \sin^4\left(\frac{i}{2}\right)\cos\left(2(\Omega-\theta-\psi)\right)\right)\Bigg] \end{aligned} \tag{14}$$

The orbit energy is written in terms of ω_n:

$$E = \frac{1}{2}\left(r^2\omega_n^2 + \dot{r}^2\right) - U(r) \tag{15}$$

When $C_{22} = 0$, $E = E_0$ $\forall t$ and the following may be written:

$$\omega_n^2 = \frac{2\left(E_0 + \tilde{U}(r)\right) - \dot{r}^2}{r^2} \tag{16}$$

where $\tilde{U}(r)$ contains only the C_{00} and C_{20} components of the gravitational potential.

Using the substitution $r(t) = r_0(1+\xi(t))$, and substituting ω_n^2 using Eqs. (14) - (16), Eq. (12) is expanded about $\xi = 0$, retaining terms quadratic in ξ for now, just to illustrate the structure:

$$\begin{aligned} \ddot{\xi} + \left(2\left(\frac{\mu}{r_0^3} + \frac{E_0}{r_0^2}\right) - 4\frac{\mu}{r_0^3}\overline{f}\right)\xi - \left(\frac{\mu}{r_0^3} + 2\frac{E_0}{r_0^2} - \frac{\mu}{r_0^3}\overline{f}\right) \\ + \left(\left(-3\frac{\mu}{r_0^3} - 2\frac{E_0}{r_0^2}\right) + 10\frac{\mu}{r_0^3}\overline{f}\right)\xi^2 + \dot{\xi}^2 = 0 \end{aligned} \tag{17}$$

where $\overline{f}$ is a function associated with the C_{20} component of the gravitational potential:

$$\overline{f}(t) = C_{20}\frac{R^2}{r_0^2}\left(\frac{3}{4}\sin^2 i_0\,(1-\cos 2\theta) - \frac{1}{2}\right) \tag{18}$$

Thus, the problem has been transformed to a study of (assumed small) variations about the initial value $r(0) = r_0$, an approximation which significantly simplifies the problem. The following change of time variables enables the subsequent non-dimensionalization of Eq. (17):

$$\tau = \left(\frac{\mu}{r_0^3}\right)^{1/2} t,\ \frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\mu}{r_0^3}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\tau} \tag{19}$$

$$
\begin{aligned}
\xi'' + &\left(2\left(1+\frac{E_0 r_0}{\mu}\right) - 4\overline{f}\right)\xi - \left(\left(1+2\frac{E_0 r_0}{\mu}\right) - \overline{f}\right) \\
&+ \left(\left(-3-2\frac{E_0 r_0}{\mu}\right) + 10\overline{f}\right)\xi^2 + \xi'^2 = 0
\end{aligned}
\tag{20}
$$

Note that $(\,)' = \mathrm{d}/\mathrm{d}\tau\,(\,)$. The change of time variables renders $\xi(\tau)$, $\xi'(\tau)$, and $\xi''(\tau)$ to all be of the same order. In this derivation, it is assumed that ξ and $\overline{f}$ are both similarly small (e.g. 10^{-2}), denoted $\mathcal{O}(\epsilon)$. This derivation could be modified to accommodate different relative scales. The smallness of ξ depends on the orbit not deviating drastically from the unperturbed geometry, and the scale of $\overline{f}$ depends on the altitude and the size of C_{20}.

Based on the assumed scales of ξ and $\overline{f}$, the $\overline{f}\xi^2$ term in Eq. (20) is significantly smaller than the others. Neglecting this term leads to the following differential equation for ξ:

$$
\xi'' + \left(2\left(1+\frac{E_0 r_0}{\mu}\right) - 4\overline{f}\right)\xi - \left(\left(1+2\frac{E_0 r_0}{\mu}\right) - \overline{f}\right) - \left(3+2\frac{E_0 r_0}{\mu}\right)\xi^2 + \xi'^2 = 0 \tag{21}
$$

This system can be initiated (without loss of generality) with $\theta_0 = 0$, then substitution of $\theta \approx n_0 t$, $n_0 = \sqrt{\mu/a_0^3}$, renders the function $\overline{f}$ as an explicit function of time t. Note that non-circular orbit angular frequency variations would appear pre-multiplied by other small terms (i.e. terms involving C_{20}), and are thus neglected. Finally, the substitution $t = \left(\mu/r_0^3\right)^{-1/2}\tau$ renders them functions of the dimensionless time: $\theta = \left(r_0^3/a_0^3\right)^{1/2}\tau$:

$$
\xi'' + \left(2\left(1+\frac{E_0 r_0}{\mu}\right) - 4\overline{f}\right)\xi - \left(\left(1+2\frac{E_0 r_0}{\mu}\right) - \overline{f}\right) - \left(3+2\frac{E_0 r_0}{\mu}\right)\xi^2 + \xi'^2 = 0 \tag{22}
$$

Identifying the small parameters ξ and $\overline{f}$ as $\mathcal{O}(\epsilon)$, the $\mathcal{O}(\epsilon)$ part of Eq. (22) is linear in ξ:

$$
\xi'' + 2\left(1+\frac{E_0 r_0}{\mu}\right)\xi = \left(1+2\frac{E_0 r_0}{\mu}\right) - \overline{f} \tag{23}
$$

To first order, $\xi(t)$ obeys simple sinusoidal dynamics with an oscillatory forcing term due to the negative of the C_{20} component of the potential. This first-order equation can be solved using the method of undetermined coefficients, noting that the harmonic forcing term has different frequencies from the homogeneous solution. The solution of Eq. (23) is the sum of the homogeneous and particular solutions given below:

$$
\xi_h(\tau) = D\cos\left(\sqrt{2\eta_1}\tau\right) + E\sin\left(\sqrt{2\eta_1}\tau\right) \tag{24}
$$

$$
\xi_p(\tau) = A\cos\left(\omega_p\tau\right) + B\sin\left(\omega_p\tau\right) + C \tag{25}
$$

where the quantities η_1 and ω_p are given:

$$
\eta_1 = 1 + \frac{E_0 r_0}{\mu},\ \omega_p = 2n_0\left(\frac{\mu}{r_0^3}\right)^{-1/2} \tag{26}
$$

Substituting the particular solution into Eq. (23), the following equations are obtained in terms of the undetermined coefficients A, B, and C:

$$
\begin{aligned}
A\left(2\eta_1 - \omega_p^2\right) &= \frac{3}{4}C_{20}\frac{R^2}{r_0^2}\sin^2 i \\
B &= 0 \\
2\eta_1 C &= 1 + 2\frac{E_0 r_0}{\mu} + \frac{1}{2}C_{20}\frac{R^2}{r_0^2}\left(1-\frac{3}{2}\sin^2 i\right)
\end{aligned}
\tag{27}
$$

Letting $\alpha = C_{20}(R/r_0)^2 \sim \mathcal{O}(\epsilon)$, the following values are obtained:

$$A = \frac{3}{4}\alpha\left(\frac{\sin^2 i}{2\eta_1 - \omega_p^2}\right),\ B = 0,\ C = \frac{1}{4}\alpha\left(\frac{1-\frac{3}{2}\sin^2 i}{\eta_1}\right) + \frac{1}{2\eta_1}\left(1 + 2\frac{E_0 r_0}{\mu}\right) \tag{28}$$

The first initial condition is $r(0) = r_0\left(1 + \xi(0)\right) = r_0$. The next initial condition on ξ is given from the following expression:

$$r'(0) = \left(\frac{\mu}{r_0^3}\right)^{-1/2}\dot{r}(0) = \left(\frac{\mu}{r_0^3}\right)^{-1/2} r_0\dot{\xi}(0) = r_0\xi'(0) \tag{29}$$

$$\xi'(0) = \left(\frac{\mu}{r_0^3}\right)^{-1/2}\frac{\dot{r}_0}{r_0} = E\sqrt{2\eta_1} \tag{30}$$

Thus, D and E are obtained from the initial conditions:

$$D = -A - C,\ E = \left(\frac{\mu}{r_0^3}\right)^{-1/2}\frac{\dot{r}_0}{r_0}\left(\frac{1}{\sqrt{2\eta_1}}\right) \tag{31}$$

The approximate solution for $\xi(\tau)$ is given by the sum of Eqs. (24) and (25) with the coefficients given in Eqs. (28) and (31), thus approximating $r(\tau) = r_0\left(1 + \xi(\tau)\right)$ to first order:

$$\xi(\tau) = A\cos\left(\omega_p\tau\right) + C + D\cos\left(\sqrt{2\eta_1}\tau\right) + E\sin\left(\sqrt{2\eta_1}\tau\right) \tag{32}$$

This simple equation is reasonably accurate for sufficiently small initial eccentricity ($e_0 \sim 10^{-3}$) and for all inclinations. Accuracy is less dependent on the osculating eccentricity, which can generally grow to larger values (10^{-2}) at some points in the orbit.

When $C_{22} \neq 0$, this procedure can still be applied for cases where the primary body is sufficiently slowly rotating ($\dot{\psi} = c \ll n$). With the slow gravity field rotation, the orbit energy will be nearly conserved on the time scale of a single orbit, and this analysis can be extended to approximate variations in $r(t)$ for several orbits. However, this scenario is somewhat rare in nature, and a less restricted solution is sought.

Approximate Solution Using Jacobi Integral for Near-Circular Orbits

With the introduction of the time-varying potential terms due to C_{22}, energy is no longer conserved in the inertial frame for this system. However, there is still a conserved quantity that can be used, existing for general uniformly rotating gravitational potentials.[2] Given a general smooth and continuous orbit potential $U(\boldsymbol{r})$, the Lagrangian is given below, along with the conjugate momenta:

$$\mathcal{L} = \frac{1}{2}\|\boldsymbol{r}' + \boldsymbol{\omega}_{B/N} \times \boldsymbol{r}\|^2 + U(\boldsymbol{r}) \tag{33}$$

$$\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{r}'} = \boldsymbol{r}' + \boldsymbol{\omega}_{B/N} \times \boldsymbol{r} \tag{34}$$

where $\boldsymbol{\omega}_{B/N}$ is the angular velocity of the primary body-fixed rotating frame and $\boldsymbol{r}'$ is the velocity seen in that frame. The Hamiltonian is given below:

$$H_J = \boldsymbol{p}\cdot\boldsymbol{r}' - \mathcal{L} = \boldsymbol{p}\cdot\left(\boldsymbol{p} - \boldsymbol{\omega}\times\boldsymbol{r}\right) - \frac{1}{2}p^2 - U \tag{35}$$

This may be written in the following form in terms of the angular momentum $\boldsymbol{L}$, and it is noted that H_J has no explicit time dependence. Thus the derivative along any orbit $\mathrm{d}H_J/\mathrm{d}t$ vanishes and H_J is thus an integral, called the Jacobi integral:[2]

$$H_J = \frac{1}{2}p^2 - U - \boldsymbol{\omega}_{B/N} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = H - \boldsymbol{\omega}_{B/N} \cdot \boldsymbol{L} \tag{36}$$

Using the celestial mechanics convention $\boldsymbol{p} = \boldsymbol{v}$ and $\boldsymbol{L} = \boldsymbol{h}$, Eq. (36) is adapted to the notation in this paper by writing $\boldsymbol{\omega}_{B/N} = c\hat{\boldsymbol{a}}_3$. Then the Jacobi integral is recognized in the following form for the second degree and order gravitational potential:[14, 15]

$$J = \frac{1}{2}v^2 - \frac{\mu}{r} - hc\cos i - U_2(r) = J_0 \tag{37}$$

where $U_2(r)$ isolates the C_{20} and C_{22} components of the gravitational potential given in Eq. (14), $h = r^2\omega_n$ is the angular momentum, and c is the primary body rotation rate. Because it has been shown that this integral exists for any uniformly rotating potential, the procedure used in this paper can in principal be extended to more complex gravitational fields.

Below, the Jacobi integral is written in terms of ω_n:

$$J = \frac{1}{2}\left(r^2\omega_n^2 + \dot{r}^2\right) - \frac{\mu}{r} - cr^2\omega_n \cos i - U_2(r) = J_0 \tag{38}$$

Re-arranging equation (38), an equation for ω_n is found in terms of r and $\dot{r}$:

$$\omega_n = c\cos i \pm \sqrt{c^2\cos^2 i - \left(\frac{\dot{r}^2}{r^2} - \frac{2}{r^2}\left(\frac{\mu}{r} + U_2(r) + J_0\right)\right)} \tag{39}$$

where the sign is negative for prograde orbits and positive for retrograde orbits. The equations are subsequently developed for prograde orbits. Eq. (39) highlights the close relationship between ω_n and c. Note that as $c \to 0$, the Jacobi integral simply becomes the orbit energy, aligning with the expectation that the conservation of energy derivation in the previous section would also be valid for very slowly-rotating bodies with $C_{22} \neq 0$. There is one additional complication for this more general case: the small variations in the inclination must be accounted for in any term that is larger than $\mathcal{O}(\epsilon)$. In this paper, that turns out to mean any term not linear in C_{20} or C_{22}. In particular, these are the $c\cos i$ terms in Eq. (39).

To obtain an expression of sufficient accuracy for the $\mathcal{O}(\epsilon)$ derivation in this paper, one may integrate Eq. (8) with the following first-order approximation:

$$i \approx i_0 + \frac{3\mu R^2}{h_0 r_0^3}\int_0^t \left(2C_{22}\sin\left(2(\Omega_0 - \psi)\right)\cos^2\theta\sin i_0 + \frac{1}{4}[C_{20} + 2C_{22}\cos\left(2(\Omega_0 - \psi)\right)]\sin 2\theta \sin 2i_0\right)\mathrm{d}t \tag{40}$$

where $\psi = ct$, by construction, from the freedom of choice in defining the arbitrary reference direction $\hat{\gamma}$. In the case of near-circular orbits, $\theta \approx \theta_0 + n_0 t$, and the result is given below:

$$
\begin{aligned}
i(t) &\approx i_0 + \frac{3\mu R^2}{4 n_0 h_0 r_0^3} C_{20} \left(\sin(\theta-\theta_0)\sin(\theta+\theta_0)\right)\sin 2i_0 \\
&+ \frac{3\mu R^2}{h_0 r_0^3} C_{22} \Bigg(\frac{1}{8c(c-n_0)(c+n_0)} \Big(-2\Big[2(c-n_0)(c+n_0)\cos 2\Omega_0 + c(c-n_0)\cos(2(\Omega_0-\theta_0)) \\
&+ (c+n_0)\big(c\cos(2(\Omega_0+\theta_0)) + 2(n_0-c)\cos(2(\Omega_0-\psi)) - c\cos(2(\Omega_0+\theta-\psi))\big) \\
&+ c(n_0-c)\cos(2(\Omega_0-\theta-\psi))\Big]\sin i_0 + c\Big[(c-n_0)\cos(2(\Omega_0-\theta_0)) - (c+n_0)\cos(2(\Omega_0+\theta_0)) \\
&+ (c+n_0)\cos(2(\Omega_0+\theta-\psi)) + (n_0-c)\cos(2(\Omega_0-\theta-\psi))\Big]\sin 2i_0\Big)\Bigg)
\end{aligned}
\tag{41}
$$

Writing this as $i(t) \approx i_0 + \delta i(t)$, for which $\delta i(t)$ is the small time-varying deviation in inclination due to the gravity field, δi^2 is assumed negligible in this derivation, and the $\cos i$ term in Eq. (39) becomes $\cos i_0 - \sin i_0 \delta i(t)$.

Substituting Eq. (39) and reusing the change of variables $r(t) = r_0(1+\xi(t))$ and non-dimensionalization of time $\tau = \left(\mu/r_0^3\right)^{1/2} t$, the following dimensionless equations are obtained from Eq. (12):

$$
\xi'' - \frac{r_0^3}{\mu}\left(\frac{\omega_n^2 r}{r_0}\right) = -\frac{1}{(1+\xi)^2} - \frac{3}{(1+\xi)^4}\overline{f}(t)
\tag{42}
$$

$$
\begin{aligned}
\frac{r_0^3}{\mu}\left(\frac{\omega_n^2 r}{r_0}\right) &= 2c^2\left(\frac{r_0^3}{\mu}\right)\left(\cos^2 i_0 - 2\cos i_0 \sin i_0 \delta i\right)(1+\xi) \\
&- c\left(\frac{r_0^3}{\mu}\right)^{1/2}\left(\cos i_0 - \sin i_0 \delta i\right)\Bigg[4c^2\left(\frac{r_0^3}{\mu}\right)\left(\cos^2 i_0 - 2\cos i_0\sin i_0\delta i\right)(1+\xi)^2 \\
&- 4\left(\xi'^2 - 2\left(\frac{1}{1+\xi} + \frac{1}{(1+\xi)^3}\overline{f} + \frac{J_0 r_0}{\mu}\right)\right)\Bigg]^{1/2} \\
&- \left[\frac{\xi'^2}{1+\xi} - \frac{2}{1+\xi}\left(\frac{1}{1+\xi} + \frac{1}{(1+\xi)^3}\overline{f} + \frac{J_0 r_0}{\mu}\right)\right]
\end{aligned}
\tag{43}
$$

where $\overline{f}$ is a function associated with the second degree and order components of the gravitational potential, now including C_{22}:

$$
\begin{aligned}
\overline{f}(t) &= C_{20}\frac{R^2}{r_0^2}\left(\frac{3}{4}\sin^2 i_0(1-\cos 2\theta) - \frac{1}{2}\right) + 3C_{22}\frac{R^2}{r_0^2}\bigg(\frac{1}{2}\sin^2 i_0 \cos(2(\Omega_0-\psi)) \\
&+ \cos^4\left(\frac{i_0}{2}\right)\cos(2(\Omega_0+\theta-\psi)) + \sin^4\left(\frac{i_0}{2}\right)\cos(2(\Omega_0-\theta-\psi))\bigg)
\end{aligned}
\tag{44}
$$

Noting that ξ and $\overline{f}$ are $\mathcal{O}(\epsilon)$, Eq. (43) is reduced to an expression that is linear in ξ and $\overline{f}$. This is done by factoring and binomial expanding the square root term and the $(1+\xi)^k$ terms. The final result for this term is reproduced below:

$$
\begin{aligned}
\frac{r_0^3}{\mu}\left(\frac{\omega_n^2 r}{r_0}\right) &\approx 2\left(\gamma_1 - \frac{\sqrt{\gamma_1}(\gamma_1-1)}{(\gamma_1+2\gamma_2)^{1/2}} - \frac{J_0 r_0}{\mu} - 2\right)\xi \\
&+ 2\left(1 + \overline{f} + \frac{J_0 r_0}{\mu} + \left((\gamma_1+2\gamma_2)^{1/2}\gamma_4 - 2\gamma_3\right)\delta i\right) \\
&+ \left(2\gamma_1 - 2\sqrt{\gamma_1}\left((\gamma_1+2\gamma_2)^{1/2}\left(1 + \frac{\overline{f}-\gamma_3\delta i}{\gamma_1+2\gamma_2}\right)\right)\right)
\end{aligned}
\tag{45}
$$

where the γ_i terms are defined below:

$$\gamma_1 = c^2 \left(\frac{r_0^3}{\mu}\right) \cos^2 i,\ \gamma_2 = 1 + \frac{J_0 r_0}{\mu},\ \gamma_3 = c^2 \left(\frac{r_0^3}{\mu}\right) \cos i_0 \sin i_0,\ \gamma_4 = \gamma_3 / \sqrt{\gamma_1} \tag{46}$$

The remaining terms in Eq. (42) are more easily simplified. The $\mathcal{O}(\epsilon)$ part of Eq. (42) is linear in ξ, and the final expression for the linear ODE is reproduced below:

$$\xi'' + 2\eta_2 \xi = 2\eta_3 - \varphi \overline{f} + \vartheta \delta i \tag{47}$$

$$\eta_2 = \gamma_2 + \sqrt{\gamma_1} \left(\frac{\gamma_1 - 1}{(\gamma_1 + 2\gamma_2)^{1/2}}\right) - \gamma_1 \tag{48}$$

$$\eta_3 = \gamma_2 - \frac{1}{2} - \sqrt{\gamma_1(\gamma_1 + 2\gamma_2)} + \gamma_1 \tag{49}$$

$$\varphi = 1 + 2\sqrt{\gamma_1} \left(\frac{1}{(\gamma_1 + 2\gamma_2)^{1/2}}\right) \tag{50}$$

$$\vartheta = 2\left(\left(\sqrt{\frac{\gamma_1}{\gamma_1 + 2\gamma_2}} - 2\right)\gamma_3 + (\gamma_1 + 2\gamma_2)^{1/2} \gamma_4\right) \tag{51}$$

Thus, Eq. (12) has been approximated by a linear constant-coefficient ODE in terms of the small parameter $\xi(\tau)$, where $r_0 \xi(t)$ represents the time-varying deviation from the orbit radius at epoch.

Eq. (47) bears some structural resemblance to Eq. (23) from the previous section. Both equations are linear oscillators with forcing terms due to the perturbations. In this case, Jacobi integral-dependent terms appear instead of energy, along with the addition of the γ_i terms. This ODE is solved in the same way as the previous section, but the new $\overline{f}$ must first be defined succinctly in terms of τ. Without loss of generality, the epoch time $t = 0$ can be defined at an instant when the first body principal axis aligns with the current line of nodes, $\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{\Omega}} = 1$. One may define $\hat{\boldsymbol{\gamma}}$ to point in this initial direction for all time, so $\Omega_0 = 0$ by construction. Then $\psi = ct$ and $\theta \approx \theta_0 + n_0 t$ render $\overline{f}$ and δi as explicit functions of time t. Other initializations are possible, but this is convenient because all initial system configurations may be captured by just two initial angles: i_0 and θ_0. Note that the resulting equations using this convention may be simplified further for equatorial orbits, so for the special case of $i_0 = 0$, there is no need for repeating the general derivation that follows.

The substitution $t = \left(\mu / r_0^3\right)^{-1/2} \tau$ renders $\overline{f}$ and δi as functions of the dimensionless time, where $\theta = \theta_0 + \left(r_0^3 / a_0^3\right)^{1/2} \tau$ and $\psi = c\left(\mu / r_0^3\right)^{-1/2} \tau$. The simplified expressions are given below, where $\alpha = C_{20} \left(R / r_0\right)^2$ and $\beta = C_{22} \left(R / r_0\right)^2$:

$$\begin{aligned}
\overline{f}(\tau) = \alpha & \left[-\frac{1}{2} + \frac{3}{4} \sin^2 i_0 (1 - \cos 2\theta_0 \cos \omega_4 \tau + \sin 2\theta_0 \sin \omega_4 \tau)\right] \\
& + 3\beta \left[\cos^4\left(\frac{i_0}{2}\right) (\cos 2\theta_0 \cos \omega_1 \tau - \sin 2\theta_0 \sin \omega_1 \tau)\right. \\
& \left. + \sin^4\left(\frac{i_0}{2}\right) (\cos 2\theta_0 \cos \omega_2 \tau - \sin 2\theta_0 \sin \omega_2 \tau) + \frac{1}{2} \sin^2 i_0 \cos \omega_3 \tau\right]
\end{aligned} \tag{52}$$

$$\begin{aligned}
\delta i(\tau) = & \frac{3}{8} \alpha \frac{\mu}{n_0 h_0 r_0} \left[\sin 2i_0 \left(\cos 2\theta_0 \left(1 - \cos \omega_4 \tau\right) + \sin 2\theta_0 \sin \omega_4 \tau\right)\right] \\
& + \frac{3}{4} \beta \frac{\mu}{c h_0 r_0} \left(\frac{1}{c^2 - n_0^2}\right) \Big[- c n_0 \sin 2i_0 \cos 2\theta_0 - 2\big(c^2 \cos 2\theta_0 + (c^2 - n_0^2)\big) \sin i_0 \\
& + \frac{1}{2} c(c + n_0) \left(\sin 2i_0 + 2 \sin i_0\right) \left(\cos 2\theta_0 \cos \omega_1 \tau - \sin 2\theta_0 \sin \omega_1 \tau\right) \\
& - \frac{1}{2} c(c - n_0) \left(\sin 2i_0 - 2 \sin i_0\right) \left(\cos 2\theta_0 \cos \omega_2 \tau - \sin 2\theta_0 \sin \omega_2 \tau\right) \\
& + 2(c^2 - n_0^2) \sin i_0 \cos \omega_3 \tau\Big]
\end{aligned} \tag{53}$$

$$\omega_1 = 2(n_0 - c)\sqrt{\frac{r_0^3}{\mu}},\ \omega_2 = 2(n_0 + c)\sqrt{\frac{r_0^3}{\mu}},\ \omega_3 = 2c\sqrt{\frac{r_0^3}{\mu}},\ \omega_4 = 2n_0\sqrt{\frac{r_0^3}{\mu}} \tag{54}$$

Eq. (54) shows that the differential equation for ξ is forced by four distinct frequencies if $n_0 \neq c$. The solution to Eq. (47) is obtained in exactly the same way as in the previous section, with the final result given below, in terms of these four forcing frequencies and the oscillator natural frequency:

$$\xi(\tau) = \sum_{i=1}^{4} A_i \cos\omega_i\tau + \sum_{i=1,\ i\neq 3}^{4} B_i \sin\omega_i\tau + C + D\cos\sqrt{2\eta_2}\tau + E\sin\sqrt{2\eta_2}\tau \tag{55}$$

$$A_1 = -3\beta\cos 2\theta_0 \left(\frac{\cos^4\left(\frac{i_0}{2}\right)\varphi - \frac{\mu}{8(c-n_0)h_0r_0}\left(\sin 2i_0 + 2\sin i_0\right)\vartheta}{2\eta_2 - \omega_1^2}\right) \tag{56}$$

$$A_2 = -3\beta\cos 2\theta_0 \left(\frac{\sin^4\left(\frac{i_0}{2}\right)\varphi + \frac{\mu}{8(c+n_0)h_0r_0}\left(\sin 2i_0 - 2\sin i_0\right)\vartheta}{2\eta_2 - \omega_2^2}\right) \tag{57}$$

$$A_3 = -\frac{3}{2}\beta\left(\frac{\sin^2 i_0\varphi - \frac{\mu}{ch_0r_0}\sin i_0\vartheta}{2\eta_2 - \omega_3^2}\right) \tag{58}$$

$$A_4 = \frac{3}{4}\alpha\cos 2\theta_0\left(\frac{\sin^2 i_0\varphi - \frac{\mu}{2n_0h_0r_0}\sin 2i_0\vartheta}{2\eta_2 - \omega_4^2}\right) \tag{59}$$

$$B_i = -\left(\frac{\sin 2\theta_0}{\cos 2\theta_0}\right)A_i,\ i = 1, 2, 4 \tag{60}$$

$$C = \frac{\alpha}{4\eta_2}\left(1 - \frac{3}{2}\sin^2 i\right)\varphi + \frac{\eta_3}{\eta_2} + \frac{3\alpha}{16}\left(\frac{\mu\sin 2i_0\cos 2\theta_0}{n_0h_0r_0\eta_2}\right)\vartheta \tag{61}$$

$$D = -\sum_{i=1}^{4} A_i - C,\ E = \frac{1}{\sqrt{2\eta_2}}\left(\sqrt{\frac{r_0^3}{\mu}}\frac{\dot{r}_0}{r_0} - \sum_{i=1,\ i\neq 3}^{4} B_i\omega_i\right) \tag{62}$$

Note the resonance condition $\omega_h^2 = 2\eta_2 = \omega_i^2$ captured by the denominators of A_i and B_i for $i = 1, 2, 3, 4$. Interestingly, these results also imply periodicity of $\xi(\tau)$, $\delta i(\tau)$, and $\overline{f}(\tau)$ if the resonance condition is avoided and $\omega_i/\omega_j \in \mathbb{Q}\ \forall i, j$ and $\omega_i/\omega_h \in \mathbb{Q}\ \forall i$, where $\mathbb{Q}$ denotes the set of rational numbers.

The behavior of the orbit radius may be approximated as $r(\tau) = r_0(1 + \xi(\tau))$ using Eq. (55) with the constants and frequencies defined above, and the transformation $\tau = \sqrt{\mu/r_0^3}t$ may be used to yield $r(t)$ explicitly. This and the previous constant-coefficient time-explicit expressions are analyzed and tested later in the paper with nonlinear truth model data.

EXPRESSIONS FOR REMAINING ORBIT PARAMETERS

With the inclination and orbit radius both approximated to $\mathcal{O}(\epsilon)$ by Eqs. (41) and (55) respectively, the approximations of variations in the other elements are now developed. Recall that in this paper, the orbit is parameterized by $\Omega, i, \theta, r, \omega_n, \dot{r}$.

Some of the variations are direct results or analogs of the previous analysis. In particular, note that $\dot{r}$ is simply approximated by $r_0\dot{\xi}$, where $\dot{\xi}$ is given below:

$$\dot{\xi}(\tau) = \sqrt{\frac{\mu}{r_0^3}}\Bigg(-\sum_{i=1}^{4} A_i\omega_i\sin\omega_i\tau + \sum_{i=1,\ i\neq 3}^{4} B_i\omega_i\cos\omega_i\tau - D\sqrt{2\eta_2}\sin\sqrt{2\eta_2}\tau$$
$$+ E\sqrt{2\eta_2}\cos\sqrt{2\eta_2}\tau\Bigg) \tag{63}$$

The angular rate ω_n is already given in Eq. (39), and can be explicitly obtained by substitution of the approximations for i, r, and $\dot{r}$ into Eq. (39), while using $\theta \approx \theta_0 + n_0 t$ in $U_2(r)$. Only $\mathcal{O}(\epsilon)$ terms should be kept for consistency with the other approximate variations. The result is given below:

$$\omega_n = c\cos i_0 - \upsilon + \left(\frac{c^2 \cos i_0 \sin i_0}{\upsilon} - c\sin i_0\right)\delta i(t) + \frac{3\mu + 2J_0 r_0}{r_0^3 \upsilon}\xi(t) - \frac{\mu}{r_0^3 \upsilon}\overline{f}(t) \tag{64}$$

where υ is a function of initial conditions:

$$\upsilon = \sqrt{\frac{2\left(\mu + J_0 r_0\right)}{r_0^3} + c^2 \cos^2 i_0} \tag{65}$$

The variation in Ω is captured to $\mathcal{O}(\epsilon)$ in the same manner as the inclination:

$$\Omega \approx \Omega_0 + \frac{3\mu R^2}{h_0 r_0^3}\int_0^t \left(C_{22}\sin\left(2(\Omega_0 - \psi)\right)\sin 2\theta + \left[C_{20} + 2C_{22}\cos\left(2(\Omega_0 - \psi)\right)\right]\cos i_0 \sin^2\theta\right) \mathrm{d}t \tag{66}$$

Reusing the free constraint $\Omega_0 = 0$ from the approximation of the orbit radius, the equation for the variation in Ω is given below for near-circular orbits:

$$\begin{aligned}\Omega(t) \approx \frac{3\mu R^2}{h_0 r_0^3}\Bigg(&C_{20}\frac{\cos i_0}{2n_0}\left(\theta - \theta_0 - \cos\theta\sin\theta + \cos\theta_0 \sin\theta_0\right) \\ &- C_{22}\frac{1}{2c(c-n_0)(c+n_0)}\Bigg(c\left(n_0 \cos 2\theta \sin 2\psi - c\left(\cos 2\psi \sin 2\theta - \sin 2\theta_0\right)\right) \\ &+ \cos i_0\left(\left(n_0^2 - c^2(1 - \cos 2\theta)\right)\sin 2\psi - c n_0\left(\cos 2\psi \sin 2\theta - \sin 2\theta_0\right)\right)\Bigg)\Bigg)\end{aligned} \tag{67}$$

This equation is evaluated by applying $\theta \approx \theta_0 + n_0 t$ and $\psi = ct$.

The argument of latitude θ is the final coordinate needed for parameterizing the orbit. Recall that the argument of latitude rate is given as $\dot{\theta} = \omega_n - \dot{\Omega}\cos i$, where the second term is due to the deviation and regression of the node from which θ is measured.[13] The approximation for $\theta(t)$ is given by integrating the following equation, substituting Eqs. (39) and (7) and retaining only terms up to $\mathcal{O}(\epsilon)$:

$$\theta(t) = \theta_0 + \int_0^t \left(\omega_n(t) - \dot{\Omega}(t)\cos i_0\right) \mathrm{d}t \tag{68}$$

Substituting preceding results into Eq. (68) and simplifying:

$$\begin{aligned}\theta(t) \approx \theta_0 + \left(c\cos i_0 - \upsilon\right)t + \left(\frac{c^2\cos i_0 \sin i_0}{\upsilon} - c\sin i_0\right)\int_0^t \delta i(t)\ \mathrm{d}t \\ + \frac{3\mu + 2J_0 r_0}{r_0^3 \upsilon}\int_0^t \xi(t)\ \mathrm{d}t - \frac{\mu}{r_0^3\upsilon}\int_0^t \overline{f}(t)\ \mathrm{d}t - \Omega(t)\cos i_0\end{aligned} \tag{69}$$

where each integral expression is given below:

$$
\begin{aligned}
\int_0^t \delta i(t)\,\mathrm{d}t = \sqrt{\frac{r_0^3}{\mu}}\Bigg(& \frac{3}{8}\alpha\frac{\mu}{n_0 h_0 r_0}\left[\sin 2i_0\left(\cos 2\theta_0\left(\tau - \frac{\sin\omega_4\tau}{\omega_4}\right) - \sin 2\theta_0\frac{\cos\omega_4\tau}{\omega_4}\right)\right] \\
& + \frac{3}{4}\beta\frac{\mu}{ch_0r_0\,(c^2-n_0^2)}\Bigg[- cn_0\sin 2i_0\cos 2\theta_0\tau - 2\big(c^2\cos 2\theta_0 + (c^2-n_0^2)\big)\sin i_0\tau \\
& + \frac{1}{2}c(c+n_0)\,(\sin 2i_0 + 2\sin i_0)\left(\cos 2\theta_0\frac{\sin\omega_1\tau}{\omega_1} + \sin 2\theta_0\frac{\cos\omega_1\tau}{\omega_1}\right) \\
& - \frac{1}{2}c(c-n_0)\,(\sin 2i_0 - 2\sin i_0)\left(\cos 2\theta_0\frac{\sin\omega_2\tau}{\omega_2} + \sin 2\theta_0\frac{\cos\omega_2\tau}{\omega_2}\right) \\
& + 2(c^2-n_0^2)\sin i_0\frac{\sin\omega_3\tau}{\omega_3}\Bigg]\Bigg)
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
\int_0^t \xi(t)\,\mathrm{d}t = \sqrt{\frac{r_0^3}{\mu}}\Bigg(& \sum_{i=1}^{4}\frac{A_i}{\omega_i}\sin\omega_i\tau - \sum_{i=1,\ i\neq 3}^{4}\frac{B_i}{\omega_i}\cos\omega_i\tau + C\tau + \frac{D}{\sqrt{2\eta_2}}\sin\sqrt{2\eta_2}\tau \\
& - \frac{E}{\sqrt{2\eta_2}}\cos\sqrt{2\eta_2}\tau\Bigg)
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
\int_0^t \overline{f}(t)\,\mathrm{d}t = \sqrt{\frac{r_0^3}{\mu}}\Bigg(& \alpha\left[-\frac{1}{2}\tau + \frac{3}{4}\sin^2 i_0(1 - \cos 2\theta_0\frac{\sin\omega_4\tau}{\omega_4} - \sin 2\theta_0\frac{\cos\omega_4\tau)}{\omega_4}\right] \\
& + 3\beta\Bigg[\cos^4\left(\frac{i_0}{2}\right)\left(\cos 2\theta_0\frac{\sin\omega_1\tau}{\omega_1} + \sin 2\theta_0\frac{\cos\omega_1\tau}{\omega_1}\right) \\
& + \sin^4\left(\frac{i_0}{2}\right)\left(\cos 2\theta_0\frac{\sin\omega_2\tau}{\omega_2} + \sin 2\theta_0\frac{\cos\omega_2\tau}{\omega_2}\right) + \frac{1}{2}\sin^2 i_0\frac{\sin\omega_3\tau}{\omega_3}\Bigg]\Bigg)
\end{aligned} \tag{72}
$$

At this point, the approximate behaviors of all 6 state elements $\Omega, i, \theta, r, \omega_n, \dot{r}$ have been developed. The necessary information for the first 5 elements are given respectively in Eqs. (67), (53), (69), (55), and (64). To use these equations, the reader is reminded of the definitions $i(\tau) = i_0 + \delta i(\tau)$, $r(\tau) = r_0(1 + \xi(\tau))$, and $\dot{r} = r_0\dot{\xi}(\tau) = r_0\sqrt{\mu/r_0^3}\xi'(\tau)$, where τ is given by Eq. (19). Differentiation of Eq. (55) is straightforward so $\xi'(\tau)$ is not explicitly given. These results enable near-circular orbits in the rotating gravity field to be analytically approximated. The elements Ω, i, θ, r capture the position, and the elements $\dot{r}$ and ω_n capture the velocity. The approximations for Ω and i are easy to obtain, so they might exist elsewhere in literature. To the knowledge of the authors, the other expressions in this paper appear here for the first time. All elements are tested numerically in this paper, but most of the focus is on studying the accuracy of the approximation of the orbit radius, whose accuracy will generally reflect the accuracy of approximations of θ and ω_n due to the coupling of these quantities.

Periodicity of the Perturbed Elements

Recall the periodicity condition on ξ, δi, and $\overline{f}$ is given as any admissible choice of angular frequencies ω_h, ω_i such that the resonance condition $\omega_h^2 = 2\eta_2 = \omega_i^2$ is avoided and $\omega_i/\omega_j \in \mathbb{Q}\ \forall i, j$ and $\omega_i/\omega_h \in \mathbb{Q}\ \forall i$. When this condition is satisfied, the analytic solutions predict that the elements $r, \dot{r}, \omega_n$, and i will be periodic. The variation of Ω has the following secular rate $\dot{\overline{\Omega}}$:

$$
\dot{\overline{\Omega}} = \frac{3\mu R^2}{2n_0h_0r_0^3}C_{20}\cos i_0 \tag{73}
$$

Common periodicity of the remaining elements θ and Ω could only be achieved by a choice of initial conditions satisfying the periodicity condition of ξ, δi, and $\overline{f}$, resulting in a common period T^* for which the elements r, $\dot{r}$, ω_n, and i are periodic, with the additional constraint that this T^* must satisfy $\theta(t_0 + T^*) = \theta(t_0) + 2\pi k$. In addition, the secular right ascension drift $\Delta\Omega$ over k orbits must be accounted for, i.e. $2\pi k = \psi + |\Delta\Omega|$, resulting in the following final constraint for bodies with $C_{20} < 0$:

$$kn_0 - \left(c - \frac{3\mu R^2}{2n_0 h_0 r_0^3} C_{20} \cos i_0\right)\left(k - \frac{1}{l}\right) = 0 \tag{74}$$

where the perturbed element common period is represented as $T^* = \left(k - \frac{1}{l}\right) T_0$ for integers k and l satisfying $k \geq 1$, $|l| > 1$. Inspection of Eq. (74) implies that for $\Gamma > 1$ and $C_{20} < 0$, the negative drift in Ω cannot be accounted for by prograde orbits, and conditions for full periodicity of the elements cannot be found. However, using the analytic results in this paper, prograde orbits that are quasi-periodic except for their precession can be found with relative ease. Such orbits could have useful applications for asteroid missions.

LIMITATIONS AND EXTENSIONS

The derivation in this paper assumes the orbit is near-circular, and that terms due to the potential (captured by $\overline{f}(t)$) and the deviations δi and $\xi(t)$ manifest at the same order in the dimensionless equations. This is not always the case, and the current treatment is inappropriate in particular for very high inclination orbits. Furthermore, the method for approximating the solution to Eq. (12) assumes that the variations in the perturbed orbit radius remain relatively small. This assumption results in poor solution accuracy near and below $\Gamma = 1$, as well as the appearance of singularities in the coefficients for $\xi(\tau)$ for when Γ is unity. Eq. (48) also breaks down at this value, so the frequency $\sqrt{2\eta_2}$ of the homogeneous solution to Eq. (47) also fails.

The failure of the current approximation at $\Gamma = 1$ is inconvenient, but can be avoided by approximating the orbital motion in a rotating coordinate system fixed to the primary body. Previous work has approximated the variation of near-circular orbits in rotating planar asymmetric potentials, in a manner that could be directly adapted for approximating perturbed motion in the GEO belt.[2] No reformulations of the solutions obtained in this paper are necessary to treat cases with the critical inclination $i_c = 63.4°$. However, for high inclinations generally greater than $70°$, there is a breakdown in approximation accuracy. The cause is due to η_2 (Eq. 48) decreasing to zero and eventually to a negative value. Because the natural frequency of $\xi(\tau)$ is $\omega_p = \sqrt{2\eta_2}$, the solution begins to fail as $\eta_2 \to 0$. The transition condition $\eta_2 = 0$ is a complicated expression, but can be approximated as $\gamma_1 = 1$ for $\Gamma = c/n_0 \gg 1$, resulting in the following inclination restriction for approximating prograde orbits under such conditions:

$$i_0 < \cos^{-1}\left(\frac{1}{c}\sqrt{\frac{\mu}{r_0^3}}\right) \tag{75}$$

Solution accuracy will start to degrade as this value is approached from below. The task of identifying and correcting the source of this failure is left to future work.

The work in this paper is focused on approximating perturbed orbits that are nearly circular. However, a possible way to alter the approach to consider deviations about more eccentric orbits would be to substitute the following ansatz into Eq. (12):

$$r(\theta) = \frac{a_0(1 - e_0^2)}{1 + e_0 \cos(\theta - \omega_0)}\left(1 + \xi(\theta)\right) \tag{76}$$

where $|\xi(\theta)| \ll 1$ is a dimensionless expression for the deviation in the radius, obtained similarly to how $\xi(\tau)$ is obtained in this paper. It describes the deviation of the orbit radius from the unperturbed elliptical orbit. The independent variable of Eq. (12) is transformed from t to θ through the following substitution:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\,) = \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(\,) = \left(\omega_n - \dot{\Omega}\cos i\right)\frac{\mathrm{d}}{\mathrm{d}\theta}(\,) \tag{77}$$

With this transformation, the approximate solution will be described in terms of the angle θ. This angle is continuously increasing and is an analog for time. Under this transformation, an additional challenge will be transforming time dependence for the angle $\psi = \psi_0 + ct$ into a function of θ.

NUMERICAL SIMULATIONS

Validating The Orbit Radius Approximations

Simulations confirm that the approximations of the orbit radius work as expected, and this section presents representative examples to demonstrate this. For all cases in this section, the hypothetical asteroid described in Table 1 is used. It is a fairly representative example of large belt asteroids.

Table 1. Primary Body Parameters

Parameter	Value
Size and Mass Data	$R = 6.0$ km, $\rho = 2.6$ g/cm^3, $m = 4.9009 \times 10^{14}$ kg
Gravity Field Data	$\mu = 3.2709 \times 10^{-5}$ km^{3}s^{-2}, $C_{00} = 1.0$, $C_{20} = -0.0903$, $C_{22} = 0.0375$
Rotation data	$10.0 < T_r < 50.0$ hours, constant variable. $\psi_0 = 0.0$

For the first simulation, $C_{22} = 0$, and the J_2-only approximation of the orbit radius is tested. The initial nonzero orbital elements are $a_0 = 40$ km, $e_0 = 0.002$, $i_0 = 50.0°$. The resulting approximation of the orbit radius is compared to truth model data in Figure 2. The results show that the approximation obtained with energy conservation works as expected with small initial eccentricity. Simulation 2 uses $a_0 = 40$ km,

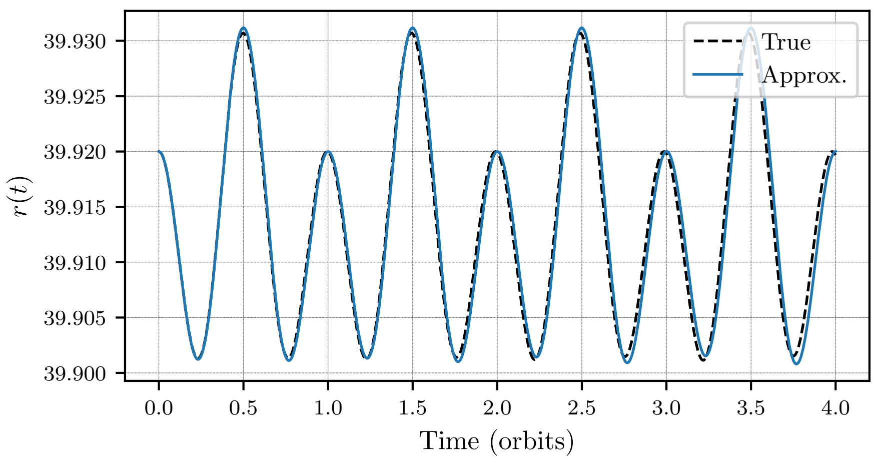

Figure 2. Orbit Radius – Simulation 1, C_{20} Only

$e_0 = 0$, and $i_0 = 2.0°, \theta_0 = 0°$. Simulation 3 uses $a_0 = 40$ km, $e_0 = 0.0022$, $i_0 = 40.0°, f_0 = \theta_0 = 50.0°$. In all cases, the unperturbed orbit period is 77.2 hours. In the first case, the asteroid rotation period is set to 24.12 hours, resulting in the angular rate ratio $\Gamma = c/n = 3.2$. For this first case, the approximation of the orbit radius is compared to truth model data in Figure 3. In the second case, the asteroid rotation period is set to 36.76 hours, so $\Gamma = 2.1$. The results are given in Figure 4.

In general, for lower values of Γ, the effects on variations in the orbit radius are more severe. In these cases of near-resonance, the fluctuations can be so large as to result in orbit ejection or impact with the primary body in the long-term. Such cases are not well-represented by any approximation assuming small deviations from the initial orbit radius, and will require a different approach. The second simulation with $\Gamma = 2.1$ is near the limit of efficacy of the current approach at this time of writing. The fluctuations are larger than in other cases, and the approximation is less accurate, while still predicting the general behavior. Overall, this approximation accuracy is limited to inclinations below 70°, as discussed earlier.

With the accuracy of the approximation of the orbit radius demonstrated for several examples, some representative results are shown to demonstrate that the other analytic approximations work. Figures 5 – 7 show

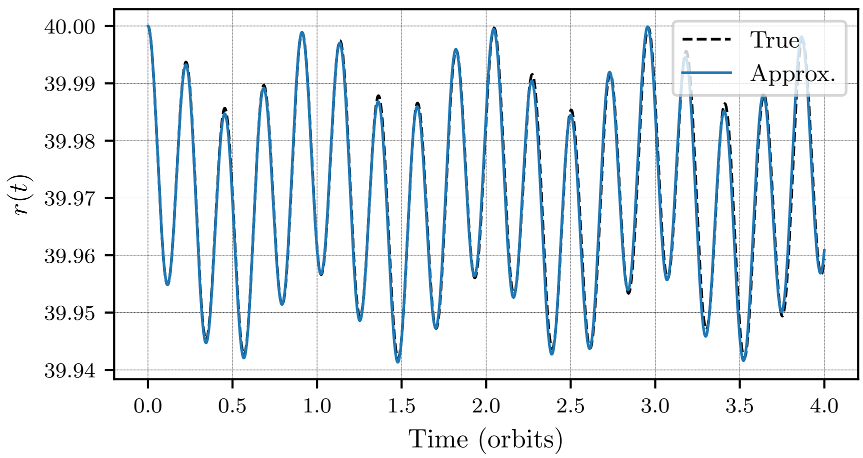

Figure 3. Orbit Radius – Simulation 2, $C_{20} + C_{22},\ \Gamma = 3.2$

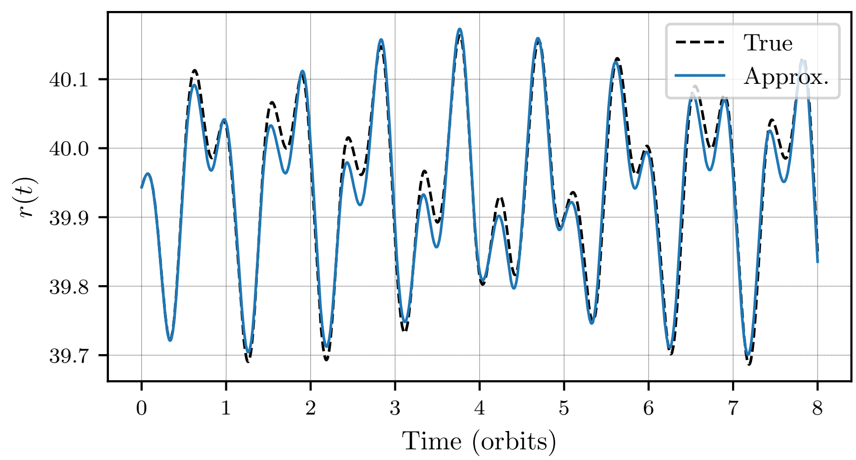

Figure 4. Orbit Radius – Simulation 3, $C_{20} + C_{22},\ \Gamma = 2.1$

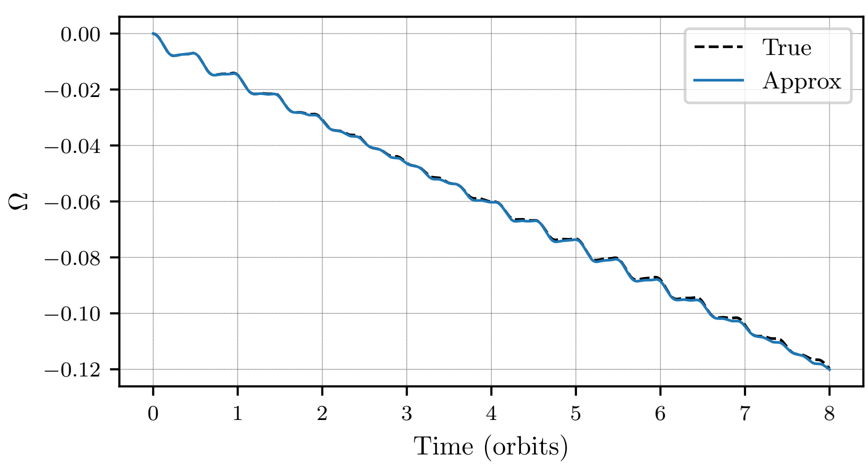

Figure 5. Right Ascension – Simulation 3, $C_{20} + C_{22},\ \Gamma = 2.1$

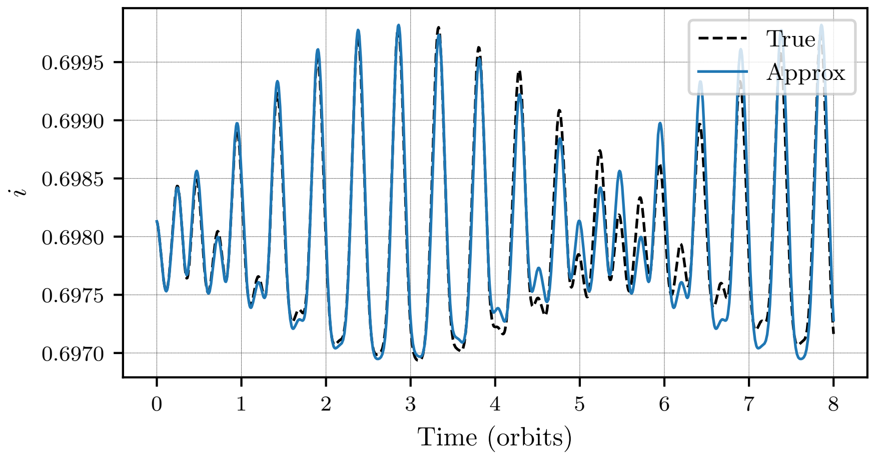

Figure 6. Inclination – Simulation 3, $C_{20} + C_{22},\ \Gamma = 2.1$

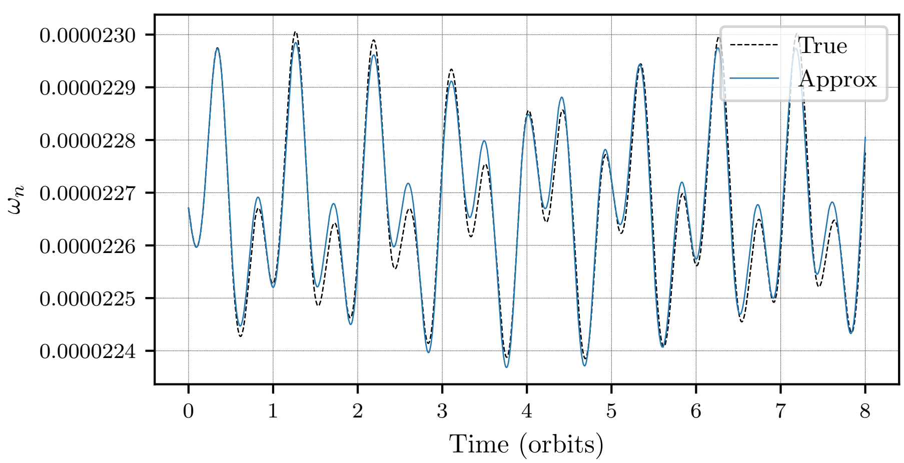

Figure 7. Angular Rate – Simulation 3, $C_{20} + C_{22},\ \Gamma = 2.1$

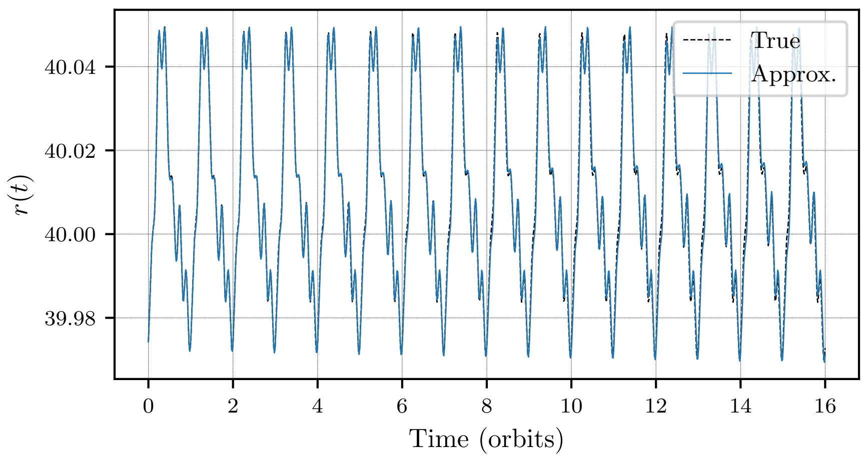

Figure 8. Orbit Radius – Simulation 4, $C_{20} + C_{22}$, $\Gamma = 4.0$

that the approximations of the right ascension, inclination, and ω_n are reasonably accurate for the case of Simulation 3, with $a_0 = 40$ km, $e_0 = 0.0022$, $i_0 = 40.0°, \theta_0 = 50.0°$. These are the same initial conditions as are used to generate Figure 4. It would be somewhat redundant to show simulation results of $\dot{r}$ and θ, so these results are omitted.

One final simulation demonstrates the efficacy of the approximation for long time spans for cases with highly regular motion. In this simulation, $a_0 = 40$ km, $e_0 = 0.001$, $i_0 = 50.0°, \omega_0 = 25.0°, f_0 = 50.0°$. Furthermore, $\Gamma = 4.0$. The resulting motion is simulated for 16 orbits with the nonlinear dynamics and with the approximation, and the results agree to high accuracy for the full timespan simulated. Only the radius data is shown for brevity, and these results are given in Figure 8. Note that the approximation captures the interesting feature of long-term variations in the brief sharp oscillations appearing $3/4$ of the way through each orbit, as well as the persistent larger orbit-periodic variations due to the initial nonzero eccentricity.

CONCLUSIONS

In this paper, the variations in the orbit radius in a rotating gravity field with C_{20} and C_{22} are described for near-circular orbits with the angular rate ratio $\Gamma = c/n > 1$. The kinematics of the osculating orbit are used to obtain a scalar differential equation for the orbit radius r, which is rendered as a time-varying differential equation in r alone using the Jacobi integral to remove unknown terms to first order in small variational terms, $\mathcal{O}(\epsilon)$. Once the approximation for the orbit radius is obtained, approximations for all other components of the orbit state are found. The approximations in this paper are all explicit functions of time. Most time-dependent terms are weighted sums of $\sin(\,)$ and $\cos(\,)$, with the weights determined by system initial conditions, and 5 fundamental frequencies constructed from the initial mean motion n_0 and the primary body angular rotation rate c. Solution accuracy generally increases as Γ is increased above the critical value $\Gamma = 1$. This makes these solutions especially well-suited for approximating near-circular orbits around quickly rotating bodies with significant C_{20} and C_{22} coefficients.

The potential for additional analytical work is extensive. First, the failure of the solution at high inclinations should be investigated and corrected. It would also be useful to construct a solution for $\Gamma < 1$, as this is the domain of low planetary orbits. The relative scale of variations in the derivation could be treated more formally. This would help in identifying which issues in the approximate solution are fundamental to this approach and which can be amended. Furthermore, a rigorous accounting of the relative scale of small terms would enable higher-order analytic series approximations of the orbit behavior to be obtained. The approximation of variations in the orbit radius can be extended to more eccentric orbits by a change of independent variable in Eq. (12). The assumption that $|\omega_n| \gg |\dot{\Omega}|, |\dot{i}|$ should also be relaxed for highly eccentric orbits.

In addition to the relative simplicity of the approximations obtained, an attractive feature of the approach in this paper is that it can be adapted to include additional terms in the gravitational potential. This new approach shows promise for refinements to develop analytic approximate solutions to the problem of orbits in complex rotating gravity fields. However, any given approximate solution will only apply to limited regions of the parameter space of all possible orbits. For example, the solutions in this paper are applicable only to orbits that remain near-circular. The assumptions used to generate an approximation necessarily constrain its applicability to the space in which these assumptions are valid. However, the parametric variation of behavior of orbital motion in rotating asymmetric gravity fields has been extensively studied numerically and with dynamical systems theory.[7,15,16] This body of work should inform where the development and application of various approximate solutions are appropriate.

ACKNOWLEDGEMENTS

This work was supported by the U.S. Department of Defense through the National Defense Science and Engineering Graduate Fellowship (NDSEG) Program. The authors thank Dr. Davide Amato for helpful comments.

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