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GRAVITATIONAL PERTURBATIONS, NONLINEARITY AND CIRCULAR ORBIT ASSUMPTION EFFECTS ON FORMATION FLYING CONTROL STRATEGIES

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Gravitational Perturbations, Nonlinearity and Circular Orbit Assumption Effects on Formation Flying Control Strategies

Kyle T. Alfriend[‡], Hanspeter Schaub[†], Dong-Woo Gim

Hill's equations have been used in most formation flying studies to determine relative motion orbits and control strategies. Hill's equations assume the Chief satellite orbit is circular, the Earth is spherically symmetric and the nonlinear terms in the relative motion variables can be neglected. This paper presents an approach for determining the effect of these assumptions on the fuel consumption for establishing and maintaining a relative motion orbit. Initial results on the errors in predicting the relative motion using Hill's equations are presented.

Nomenclature

Subscripts

c – refers to Chief satellite

d – refers to deputy satellite

0 – refers to conditions at the initial time

Reference frames

E – Earth centered inertial

C – chief orbit frame with *x*-axis along the radius vector, the *y*-axis in the direction of motion and the *z*-axis perpendicular to the orbit plane. Origin coincides with chief satellite. Unit vectors are $\left(\vec{e}_{xc}, \vec{e}_{yc}, \vec{e}_{zc}\right)$.

D – deputy orbit frame with *u*-axis along the radius vector, the *v*-axis in the direction of motion and the *w*-axis perpendicular to the orbit plane. Origin coincides with deputy satellite. Unit vectors are $\left(\vec{e}_{xd}, \vec{e}_{yd}, \vec{e}_{zd}\right)$.

Variables

T^{BA} - transformation matrix for transforming a vector from the *A* frame to the *B* frame.

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$\left(\vec{R}_c, \vec{R}_d\right)$- position vectors of the chief and deputy.

$\vec{r}_d$ - position of the deputy relative to the chief.

$\left(\vec{V}_c, \vec{V}_d\right)$ - velocity of the Chief and deputy.

$\left(V_r, V_t\right)$ - radial and tangential components of the velocity.

$\vec{v}_d$ - velocity of the deputy relative to the chief.

$\left(v_{dr}, v_{dt}\right)$ - radial and tangential components of the relative velocity vector.

a – semi-major axis

e – eccentricity

i – inclination

Ω – right ascension

ω – argument of perigee.

f – true anomaly

θ – argument of latitude, $\theta = f + \omega$

q_1 – $e\cos\omega$

q_2 – $e\sin\omega$

$\delta\alpha$ - variation of the variable α with respect to the chief orbit.

INTRODUCTION

Spacecraft flying in precise formation is a subject drawing considerable attention within NASA and the DoD [1]. O-orbit experiments are planned within the near future [2,3]. Satellites flying in formation is not a new challenge, but flying in precise formation and operating autonomously is a significant challenge. It is important to design the relative motion orbits such that fuel consumption is minimized and lifetime maximized. Most studies [4-6] have used Hill's equations [7] (sometimes called the Clohessy-Wiltshire or CW equations) to describe the relative motion of the satellites. These equations assume that a) the Earth is spherically symmetric, b) the Chief or reference satellite orbit is circular, and c) the equations can be linearized in the relative motion variables. For small formations the effects of the neglected nonlinear terms are probably negligible, but the effects of the ignored gravitational perturbations and the eccentric reference orbit can be significant[7-9]. Formations that will emulate large apertures will require some of the satellites to have out-of-plane motion relative to the reference orbit. This out-of-plane motion is achieved by some combination of small changes in the inclination, δi, and the right ascension, $\delta\Omega$. (see Figure 1) An inclination difference results in the maximum out-of-plane separation occurring at the maximum latitude. In contrast, a right ascension difference results in the maximum separation occurring at the equator. A constellation emulating a large aperture at all times would have satellites with varying combinations of inclination and right ascension differences. A differential inclination has three negative effects, it causes the deputy satellites to have a slightly different nodal precession rate, a slightly different orbit period and a slightly different argument of perigee rate. Since all three of these effects cause the two satellites to slowly separate these effects must be negated, either by control or design of the relative motion orbit. Reference 7 derives initial conditions for minimizing these effects by selection of the

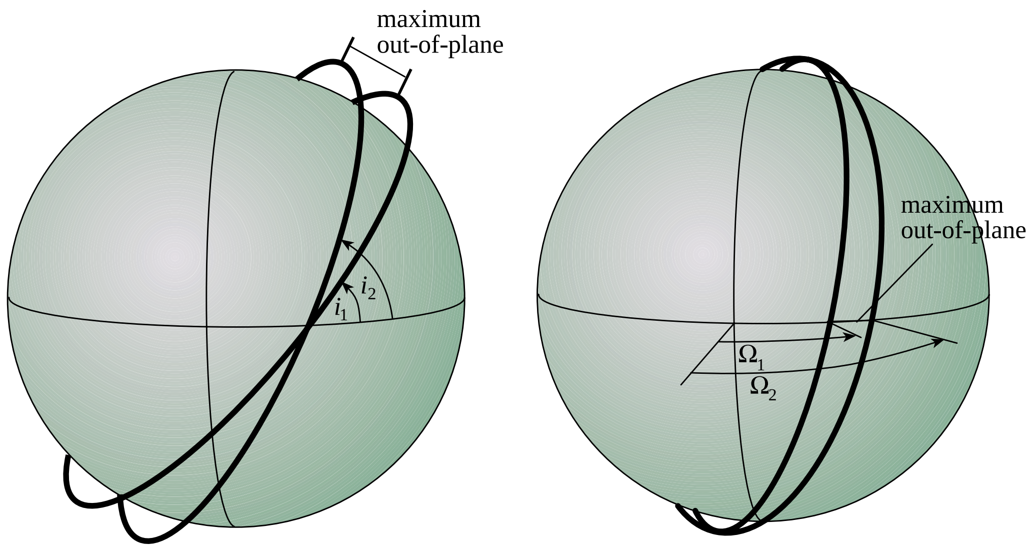


Figure 1 Achieving out-of-plane motion

orbital parameters. However, in most cases some control will still be needed. Some control approaches are presented in references 4,6,8 and 9. An unanswered question is what effect does using Hill's equations for the control have on fuel consumption. The gravitational perturbations create short period oscillations in the orbital elements that then create short period oscillations in the relative motion variables that are not captured by Hill's equations. If these oscillations are outside of the deadband of the control system they cold be interpreted by the control system as a secular rate and the control system would then try to negate these natural motions. The reference orbit eccentricity can have a similar effect. Trying to continually negate these natural motions will waste fuel. How much is the unanswered question. Also, if the model does not include the gravitational perturbation effects then fuel may be wasted trying to negate the differential secular rates. The system will not know how to select the correct orbital parameters to minimize the secular rates. The purpose of this paper is to develop a method for evaluating these effects and to present some results on the errors that occur in estimating the relative motion with Hill's equations. Essentially, we develop a state transition matrix for a system that includes the gravitational perturbations and reference orbit eccentricity. Research is underway on another method for a state transition matrix that will include the effect of the neglected nonlinear terms. One of the methods includes the effects of the nonlinear terms in the relative motion variables. A state transition matrix that includes the reference orbit eccentricity for small eccentricities has been derived by Melton [10].

HILL'S EQUATIONS

Referring to figure 2 the relative motion is described using reference frame *O*, which is a rotating reference with its origin at the reference satellite, the *x*-axis is along the radius vector, the *y*-axis is in the orbit plane in the direction of motion, and the *z*-axis is perpendicular to the orbit plane. Assuming the reference satellite orbit is circular and there are no gravitational perturbations the linearized relative equations of motion are:

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2 x = 0 \\
&\ddot{y} + 2n\dot{x} = 0 \\
&\ddot{z} + n^2 z = 0 \\
&n = \text{mean motion of reference orbit}
\end{aligned} \tag{1}$$

Note that the out-of-plane motion is decoupled from the in-plane motion. The solution is

$$\begin{aligned}
&x = 2(2x_0 + \dot{y}_0 / n) - (3x_0 + 2\dot{y}_0 / n)\cos\psi + (\dot{x}_0 / n)\sin\psi \\
&y = (y_0 - 2\dot{x}_0 / n) - 3(2x_0 + \dot{y}_0 / n)\psi + (2\dot{x}_0 / n)\cos\psi + 2(3x_0 + 2\dot{y}_0 / n)\sin\psi \\
&z = z_0 \cos\psi + (\dot{z}_0 / n)\sin\psi \\
&\psi = nt
\end{aligned} \tag{2}$$

For periodic motion

$$2x_0 + \dot{y}_0 / n = 0 \tag{3}$$

This condition is just the requirement that the semi-major axes of the two satellites must be equal. Also requiring that the center of the relative motion be at the reference satellite periodic relative motion orbits are given by

$$\begin{aligned}
&x = x_0 \cos\psi + (y_0 / 2)\sin\psi = A\sin(\psi + \alpha) \\
&y = y_0 \cos\psi - 2x_0 \sin\psi = 2A\cos(\psi + \alpha) \\
&z = z_0 \cos\psi + (\dot{z}_0 / n)\sin\psi = B\sin(\psi + \beta) \\
&A = \left(x_0^2 + y_0^2 / 4\right)^{1/2}, \tan\alpha = 2x_0 / y_0 \\
&B = \left(z_0^2 + \dot{z}_0^2 / n^2\right)^{1/2}, \tan\beta = nz_0 / \dot{z}_0
\end{aligned} \tag{4}$$

Note that the projection of the relative periodic orbits in the *x*-*y* (orbit) plane is a 2-1 with the long axis in the *y*-direction. Two periodic orbits of interest are a) a circular relative motion orbit, and b) an orbit for which the projection of the motion in the horizontal (*y*-*z*) plane is a circle. This has application for emulating large circular apertures. The initial conditions for these orbits are:

Circular Relative Orbit

$$B = \sqrt{3}A, \alpha = \beta$$
$$x^2 + y^2 + z^2 = 4A^2 \tag{5}$$

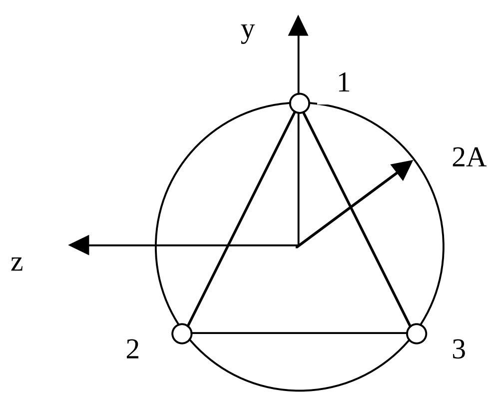
This relative motion orbit is inclined at 30 degrees to the horizontal plane.

Circular Horizontal Plane Orbit

$$B = 2A, \alpha = \beta$$
$$y^2 + z^2 = 4A^2 \tag{6}$$

This relative motion orbit is inclined at 26.56 degrees to the horizontal plane.

The TechSat21 program has three satellites flying in formation. One option for a portion of the program is the circular horizontal plane orbit. In this configuration the three satellites would form an equilateral triangle in the horizontal plane. Thus, the constellation would appear to be a rotating equilateral triangle. For this configuration the differential inclination will be different for each satellite. Thus, the rate that each satellite drifts from the equilateral triangle configuration will be different. Assume that at *t=0* the chief satellite is on the equator and the constellation is as shown in Figure 2 and is rotating counterclockwise as a circle of radius 2*A*. The initial conditions for the three satellites are



$$y_{10} = 2A, \dot{y}_{10} = 0, z_{10} = 0, \dot{z}_{10} = 2An, \alpha_1 = 0$$
$$y_{20} = -A, \dot{y}_{20} = -\sqrt{3}An, z_{20} = \sqrt{3}A, \dot{z}_{20} = -An, \alpha_2 = 120\,\deg \tag{7}$$
$$y_{30} = -A, \dot{y}_{30} = \sqrt{3}An, z_{30} = -\sqrt{3}A, \dot{z}_{30} = -An, \alpha_2 = -120\,\deg$$

The change in inclination and right ascension to achieve this desired motion when the chief satellite is on the equator is

$$\delta i = \dot{z}_0 / Rn, \delta\Omega = -z_0 R \tag{8}$$

where R is the radius of the chief's orbit. Thus, the inclination and right ascension changes for the three satellites are

$$\begin{aligned} \delta i_1 &= 2A/R, \delta\Omega_1 = 0 \\ \delta i_2 &= -A/R, \delta\Omega_2 = -\sqrt{3}A/R \\ \delta i_3 &= -A/R, \delta\Omega_3 = \sqrt{3}A/R \end{aligned} \tag{9}$$

Gravitational Perturbation Effects

The primary gravitational perturbation effect is due to the equatorial bulge term, J_2. The J_2 term changes the orbit period, a drift in perigee, a nodal precession rate and periodic variations in all the elements. Let's consider the right ascension rate which is

$$\dot{\Omega} = -\frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 n \cos i \tag{10}$$

If a change in inclination is used to create out-of-plane motion a differential nodal precession rate occurs which causes the planes to slowly separate. The differential rate is

$$\delta\dot{\Omega} = -\dot{\Omega}_c \tan i \,\delta i \tag{11}$$

Consider the case when the out-of-plane motion is caused by only an inclination change. This means that at equator crossings there is no out-of-plane separation. Letting $\rho = 2A$ be the radius of the relative motion orbit the growth in the out-of-plane separation at the equator is

$$\frac{\delta\rho_i}{\rho}(per\ day) = 0.118 \tag{12}$$

Thus, the circle begins to distort. The rate of distortion is a function of the changes in inclination and right ascension used to create the out-of-plane motion, thus it would be different for each satellite. The question is what is the effect on fuel consumption and system performance of using Hill's equations for the control system model and continuing to let the control system correct this growth. Including these effects in the design of the relative motion orbits can minimize this growth[7] and including them in the control system model may improve fuel consumption and reduce the frequency of control. In addition to these secular out-of-plane effects there are in-plane secular effects and the effects of the short period variations due to J_2 and the orbit eccentricity.

Developed in this paper are two analytic methods that can be used to evaluate the effects of neglecting these terms in the orbit design and control.

PROBLEM FORMULATION

To evaluate the effect of neglecting the chief satellite eccentricity, the gravitational perturbations and nonlinearities the control needs to be evaluated with and without these effects included in the model. Thus, the objective is to obtain a state transition matrix with these effects included. In this paper it will be assumed that the eccentricity is small, basically it will be assumed that $e = O(J_2)$. A state transition matrix for small eccentricity has been obtained by Melton [10]. There are two approaches for obtaining the state transition matrix. One approach is to use the geometry of the problem and realize that the deputy relative motion is a result of small changes in the chief satellite orbital elements. This method will be called the geometric method and is developed in this paper. A second approach is to write the equations of motion in the rotating reference frame, but include the gravitational perturbations and not assume a circular orbit for the chief satellite. In addition, the non-linear terms in the relative motion variables can be included. A solution using a perturbation method can then be used to obtain a solution. There are several perturbation methods that can be used to obtain the problem. A perturbation solution of this problem using Hamiltonian mechanics and Lie Series is currently underway and will be reported on in a later publication. The primary reasons for using this approach are:

- The large amount of algebraic manipulations required are easily implemented on the computer.
- The solution will be in a form to easily evaluate what effects need to be included in the state transition matrix. For example, do the periodic effects due to J_2 need to be included or is it sufficient to just include the secular effects.

This method will be referred to as the Hamiltonian method.

Geometric Method

Let

$$\begin{aligned} \mathbf{x}^T &= (x, \dot{x}, y, \dot{y}, z, \dot{z}) \\ \mathbf{e}^T &= (a, \theta, i, q_1, q_2, \Omega) \end{aligned} \tag{13}$$

where θ is the argument of latitude and $q_1 = e\cos\omega, q_2 = e\sin\omega$. These variables are used because the true anomaly and argument of perigee are undefined for zero eccentricity. Since the relative motion is small the approach will be to express the orbital elements of the deputy as a Taylor series about the chief satellite elements. Thus,

$$\mathbf{e}_d = \mathbf{e}_c + \delta\mathbf{e} \tag{14}$$

W now want to relate the $\delta\mathbf{e}$ to the relative motion state $\mathbf{x}$. The deputy's position is

$$\vec{R}_d = \vec{R}_c + \vec{\rho} = (R+x)\vec{e}_{xc} + y\vec{e}_{yc} + z\vec{e}_{zc} \tag{15}$$

The deputy's position in the chief reference frame is also given by

$$\vec{R}_d^C = T^{CE} T^{ED} \vec{R}_d^D \tag{16}$$

The transformation from the orbit frame O (C or D) to the inertial frame E is given by

$$\begin{gathered} T^{OE} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ T^{OE} = \begin{pmatrix} c\theta c\Omega - s\theta c i s\Omega & c\theta s\Omega + s\theta c i c\Omega & s\theta s i \\ -s\theta c\Omega - c\theta c i s\Omega & -s\theta s\Omega + c\theta c i c\Omega & c\theta s i \\ s i s\Omega & -s i c\Omega & c i \end{pmatrix} \end{gathered} \tag{17}$$

Some identities that will be used are

$$\begin{gathered} e\cos f = e\cos(\theta-\omega) = q_1\cos\theta + q_2\sin\theta \\ e\sin f = e\sin(\theta-\omega) = q_1\sin\theta - q_2\cos\theta \\ V_t = R\dot{\theta} \\ V_r = \dot{R} = \frac{R^2\dot{\theta}}{p} e\sin f = \frac{h}{p}\left[q_1\sin\theta - q_2\cos\theta\right], p = a\left(1-e^2\right) \\ \dot{\theta} = \frac{h}{R^2} = \sqrt{\frac{\mu}{p}}(1+e\cos f)^2 \\ \dot{\theta} = an\eta^{-1}\left(1+q_1\cos\theta + q_2\sin\theta\right)^2 \end{gathered} \tag{18}$$

Now expand eq. (16) about the chief satellite motion.

$$\begin{gathered} \vec{R}_d^C = T^{CE}\left(T^{EC} + \delta T^{EC}\right)\begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} = \left(I + T^{CE}\delta T^{EC}\right)\begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} \\ \vec{R}_d^C = \begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} + R_c T^{CE}\begin{pmatrix} \delta T_{11} \\ \delta T_{12} \\ \delta T_{13} \end{pmatrix} \end{gathered} \tag{19}$$

The δT_{ij} are

$$
\begin{aligned}
\delta T_{11} &= T_{21c}\delta\theta - T_{12c}\delta\Omega + (T_{13c}\sin\Omega_c)\delta i \\
\delta T_{12} &= T_{22c}\delta\theta + T_{11c}\delta\Omega - (T_{13c}\cos\Omega_c)\delta i \\
\delta T_{13} &= T_{23c}\delta\theta + \left(\sin\theta_c \cos i_c\right)\delta i
\end{aligned}
\tag{20}
$$

Substitute eq. (20) into eq. (19).

$$
\begin{aligned}
\vec{R}_d^C &= \begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} + R_c T_{C2E} \begin{pmatrix} T_{21c}\delta\theta - T_{12c}\delta\Omega + (T_{13c}\sin\Omega_c)\delta i \\ T_{22c}\delta\theta + T_{11c}\delta\Omega - (T_{13c}\cos\Omega_c)\delta i \\ T_{23c}\delta\theta + \left(\sin\theta_c \cos i_c\right)\delta i \end{pmatrix} \\
\vec{R}_d^C &= \begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} + R_c \begin{pmatrix} 0 \\ \delta\theta + \delta\Omega\cos i_c \\ -\cos\theta_c \sin i_c \delta\Omega + \sin\theta_c \delta i \end{pmatrix}
\end{aligned}
\tag{21}
$$

Thus,

$$
\begin{aligned}
x &= \delta R_c \\
y &= R_c\left(\delta\theta + \delta\Omega\cos i_c\right) \\
z &= R_c\left(-\cos\theta_c \sin i_c \delta\Omega + \sin i_c \delta i\right)
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
R &= \frac{a\left(1-e^2\right)}{1+e\cos f} = \frac{a\left(1-q_1^2-q_2^2\right)}{1+q_1\cos\theta + q_2\sin\theta} \\
\delta R &= \frac{R_c}{a_c}\delta a - 2R_c\left(\frac{a_c}{p_c}\right)\left(q_{1c}\delta q_1\right) - \frac{R_c^2}{p_c}\left[\left(-q_{1c}\sin\theta + q_{2c}\cos\theta_c\right)\delta\theta + \delta q_1\cos\theta_c + \delta q_{2c}\sin\theta_c\right]
\end{aligned}
\tag{23}
$$

The deputy’s velocity in the chief reference frame is

$$
\vec{V}_d^C = \left(V_{rc} + \dot{x} - \dot{\theta}_c y\right)\vec{e}_x + \left(V_{tc} + \dot{y} + \dot{\theta}_c x\right)\vec{e}_y + \dot{z}\vec{e}_z
\tag{24}
$$

Also,

$$\vec{V}_d^C = T^{CE} T^{ED} \begin{pmatrix} V_{rd} \\ V_{td} \\ 0 \end{pmatrix} = T^{CE}\left(T^{EC} + \delta T^{CE}\right) \begin{pmatrix} V_{rc} + \delta V_r \\ V_{tc} + \delta V_t \\ 0 \end{pmatrix}$$

$$\vec{V}_d^C = \begin{pmatrix} V_{rc} + \delta V_{rc} \\ V_{tc} + \delta V_{tc} \\ 0 \end{pmatrix} + T^{CE} \begin{pmatrix} V_{rc}\delta T_{11} + V_{tc}\delta T_{21} \\ V_{rc}\delta T_{12} + V_{tc}\delta T_{22} \\ V_{rc}\delta T_{13} + V_{tc}\delta T_{23} \end{pmatrix} \tag{25}$$

$$\begin{aligned} \delta T_{21} &= -T_{11c}\delta\theta - T_{22c}\delta\Omega + T_{23c}\sin\Omega_c \delta i \\ \delta T_{22} &= -T_{12c}\delta\theta + T_{21c}\delta\Omega - T_{23c}\cos\Omega_c \delta i \\ \delta T_{23} &= -T_{13c}\delta\theta + \cos\theta_c \cos i_c \delta i \end{aligned} \tag{26}$$

Substituting gives

$$\vec{V}_d^C = \begin{pmatrix} V_{rc} + \delta V_{rc} \\ V_{tc} + \delta V_{tc} \\ 0 \end{pmatrix} + V_{rc} \begin{pmatrix} 0 \\ \delta\theta + \delta\Omega \cos i_c \\ -\cos\theta_c \sin i_c \delta\Omega + \sin\theta_c \delta i \end{pmatrix} + V_{tc} \begin{pmatrix} -\delta\theta - \cos i_c \delta\Omega \\ 0 \\ \sin\theta_c \sin i_c \delta\Omega + \cos\theta_c \delta i \end{pmatrix} \tag{27}$$

$$\begin{aligned} V_{tc} &= R_c\dot{\theta}_c = \frac{h_c}{R_c} = \frac{\sqrt{\mu p_c}}{R_c} = \sqrt{\frac{\mu}{p_c}}\left(1 + e_c \cos f_c\right) = \sqrt{\frac{\mu}{p_c}}\left(1 + q_{1c}\cos\theta_c + q_{2c}\sin\theta_c\right) \\ \delta V_t &= -\frac{V_{tc}}{2p_c}\delta p + \sqrt{\frac{\mu}{p_c}}\left[\delta q_1 \cos\theta_c + \delta q_2 \sin\theta_c + \left(-q_{1c}\sin\theta_c + q_{2c}\cos\theta_c\right)\delta\theta\right] \end{aligned} \tag{28}$$

The variation in the radial velocity is

$$\begin{aligned} V_{rc} &= \dot{R}_c = \sqrt{\frac{\mu}{p_c}}\left(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c\right) \\ \delta V_r &= -\frac{1}{2p_c}V_{rc}\delta p + \sqrt{\frac{\mu}{p_c}}\left[\left(\delta q_1 \sin\theta_c - \delta q_2 \cos\theta_c\right) + \left(q_{1c}\cos\theta_c + q_{2c}\sin\theta_c\right)\delta\theta\right] \end{aligned} \tag{29}$$

Substituting for δp gives

$$\delta V_t = -\frac{V_{tc}}{2a_c}\delta a + \sqrt{\frac{\mu}{p_c}}\left(-q_{1c}\sin\theta_c + q_{2c}\cos\theta_c\right)\delta\theta$$
$$+\left(\frac{V_{tc}a_c q_{1c}}{p_c} + \sqrt{\frac{\mu}{p_c}}\cos\theta_c\right)\delta q_1 + \left(\frac{V_{tc}a_c q_{2c}}{p_c} + \sqrt{\frac{\mu}{p_c}}\sin\theta_c\right)\delta q_2 \tag{29}$$

$$\delta V_r = -\frac{1}{2p_c}V_{rc}\left[\left(\frac{p_c}{a_c}\right)\delta a - a_c\left(2q_{1c}\delta q_1 + 2q_{2c}\delta q_2\right)\right]$$
$$+\sqrt{\frac{\mu}{p_c}}\left[\left(\delta q_1 \sin\theta_c - \delta q_2\cos\theta_c\right) + \left(q_{1c}\cos\theta_c + q_{2c}\sin\theta_c\right)\delta\theta\right] \tag{30}$$

The velocity development is now complete. Using eq. (22b) we get

$$\begin{aligned}
\dot{x} &= \delta V_r \\
\dot{y} + \dot{\theta}_c x - \left(V_{rc} / R_c\right)y &= \delta V_t \\
\dot{z} &= \left(V_{tc}\cos\theta_c + V_{rc}\sin\theta_c\right)\delta i + \left(V_{tc}\sin\theta_c - V_{rc}\cos\theta_c\right)\sin i_c \delta\Omega
\end{aligned} \tag{31}$$

This completes the development. We now have

$$\mathbf{x} = A\delta\mathbf{e} \tag{32}$$

The elements of A and its inverse A^{-1} are given in the Appendix. Please note that we have made no restrictions on the orbital elements. They can be two body elements or they can be the solution from an analytic theory such as Brouwer's theory [11]. Whatever theory is used we can develop

$$\begin{aligned}
\delta\mathbf{e}(t) &= \Phi_{\mathbf{e}}(t)\delta\mathbf{e}(t_0) \\
\delta\mathbf{e}_0 &= A^{-1}(t_0)\mathbf{x}(t_0)
\end{aligned} \tag{33}$$

giving

$$\begin{aligned}
\mathbf{x}(t) &= \Phi_{\mathbf{x}}(t)\mathbf{x}_0 \\
\Phi_{\mathbf{x}}(t) &= A(t)\Phi_{\mathbf{e}}(t)A^{-1}(t_0)
\end{aligned} \tag{34}$$

The development is now shown for a circular chief orbit. Since there are no perturbations the only time varying element is the argument of latitude. Using the angular momentum integral

$$r^2 d\theta = \sqrt{\mu p}\,dt$$
$$\sqrt{\frac{p^3}{\mu}}\frac{d\theta}{\left(1+q_1\cos\theta+q_2\sin\theta\right)^2}=dt \tag{35}$$

Now integrate

$$\sqrt{\frac{p^3}{\mu}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1+q_1\cos\theta+q_2\sin\theta\right)^2}=t \tag{36}$$

Expand this equation in a Taylor series about the chief orbit.

$$\begin{aligned}0=1.5\frac{\delta p}{p_c}\sqrt{\frac{p_c^3}{\mu}}\int_{\theta_{0c}}^{\theta_c}\frac{d\tau}{\left(1+q_{1c}\cos\tau+q_{2c}\sin\tau\right)^2}-2\sqrt{\frac{p_c^3}{\mu}}\int_{\theta_0}^{\theta}\frac{\left(\delta q_1\cos\tau+\delta q_2\sin\tau\right)d\tau}{\left(1+q_{1c}\cos\tau+q_{2c}\sin\tau\right)^3}\\ +\delta\theta\sqrt{\frac{p_c^3}{\mu}}\frac{1}{\left(1+q_{1c}\cos\theta_c+q_{2c}\sin\theta_c\right)^2}-\delta\theta_0\sqrt{\frac{p_c^3}{\mu}}\frac{1}{\left(1+q_{1c}\cos\theta_{0c}+q_{2c}\sin\theta_{0c}\right)^2}\end{aligned} \tag{37}$$

Now set e=0 and use eq. (42).

$$\delta\theta=\delta\theta_0-1.5n_c\frac{\delta a}{a_c}+2\delta q_1\sin\theta_c+\delta q_2\left(1-\cos\theta_c\right) \tag{38}$$

Therefore,

$$\Phi_{\mathbf{e}}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1.5 n_c t / a_c & 1 & 0 & 2\sin\theta_c & 2(1-\cos\theta_c) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

RESULTS

To evaluate how accurately the method developed here estimates the relative motion and to determine the errors resulting from using Hill's equations the following example was used. The initial conditions used for the deputy result in the Circular Horizontal plane orbit when the Earth is spherically symmetric and the chief satellite orbit is circular.

Chief orbital elements	Deputy initial conditions and elements	
$a_c = 7100\,km$	$x = 0$	$\delta a = 0$
$e_c = 0.005$	$\dot{x} = n_c / 4$	$\delta\theta = 0.004055\,\deg$
$i_c = 70\,\deg$	$y = 0.5\,km$	$\delta i = 0.0040148\,\deg$
$\Omega_c = \omega = f = 0$	$\dot{y} = 0$	$\delta q_1 = 0$
$\theta_c = 0$	$z = 0$	$\delta q_2 = -3.556\times10^{-5}$
$q_{1c} = 0.005$ $q_{2c} = 0$	$\dot{z} = n/2$	$\delta\Omega = 0$
$n = \sqrt{\dfrac{\mu}{a^3}} = 1.05531\times10^{-3}\, r/s$		

The method is first compared to Hill's equations and the exact solution for a spherically symmetric Earth. Figure 3 shows the error in the new method for one day. The errors are only several centimeters. Figure 4 shows the errors resulting from the use of Hill's equations. The errors in Hill's equations are considerably larger than the geometric method. For these results to obtain the mean motion in Hill's equation we used the mean semi- The periodic variations result from the circular orbit assumption and the secular growth in the in-track direction is mostly due an incorrect mean motion.

Figure 5a shows the relative motion trajectory with the gravitational perturbations included (J_2-J_4) and Figure 5b show the same trajectory using the Geometric Method. Obviously the Geometric Method is incorporating the primary eccentricity and gravitational perturbation effects.

Figures 6 and 7 show the errors that occur when estimating the motion with the Geometric Method and Hill's equations, respectively. The Geometric results in much smaller errors. The in-track error of 20 m after one day is the result of approximately a 2m error in semi-major axis. In

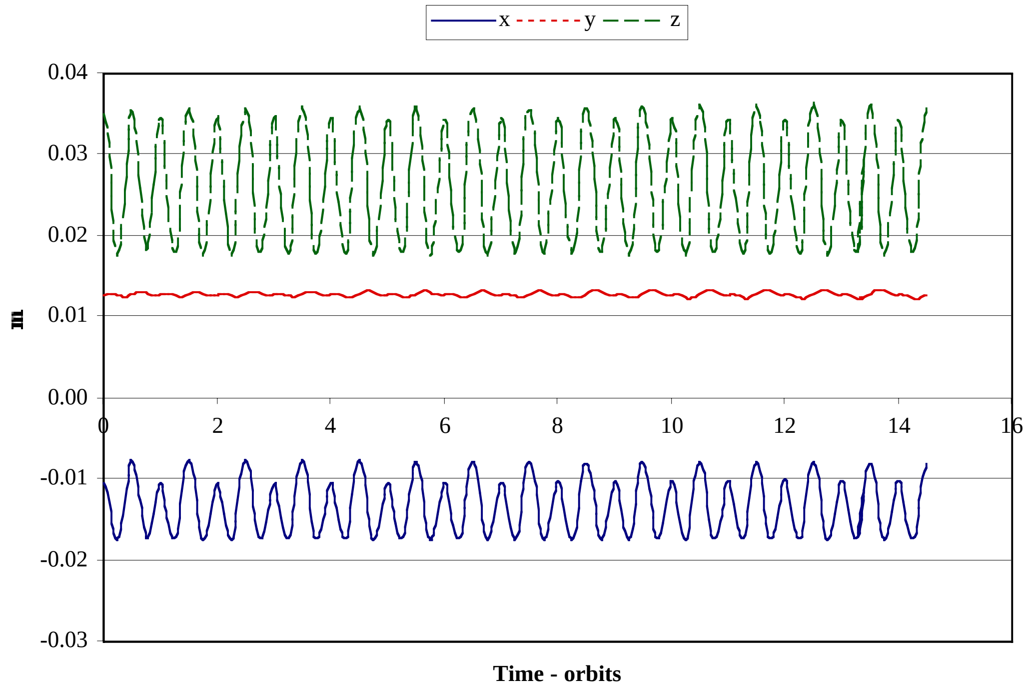


Figure 3 Geometric Method Errors for Spherically Symmetric Earth

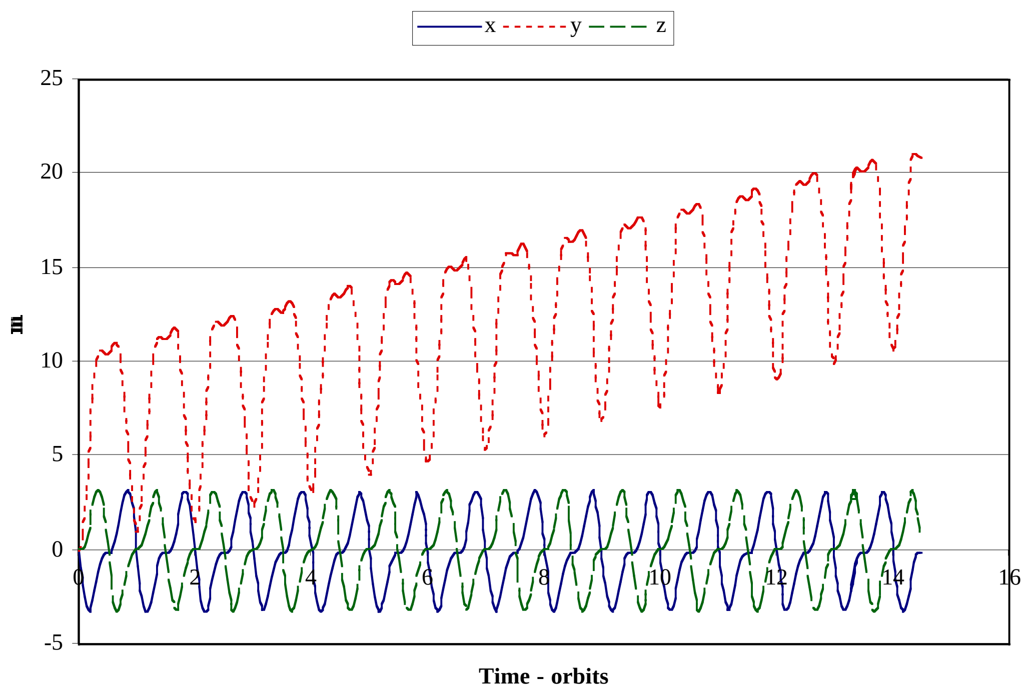


Figure 4 Hill's Equations Errors for Spherically Symmetric Earth

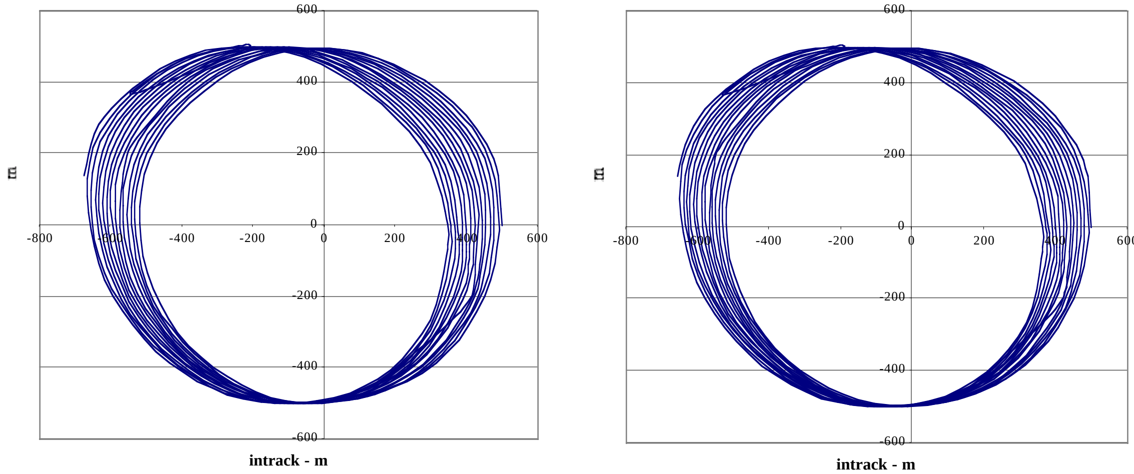


Figure 5 Horizontal Plane Trajectory

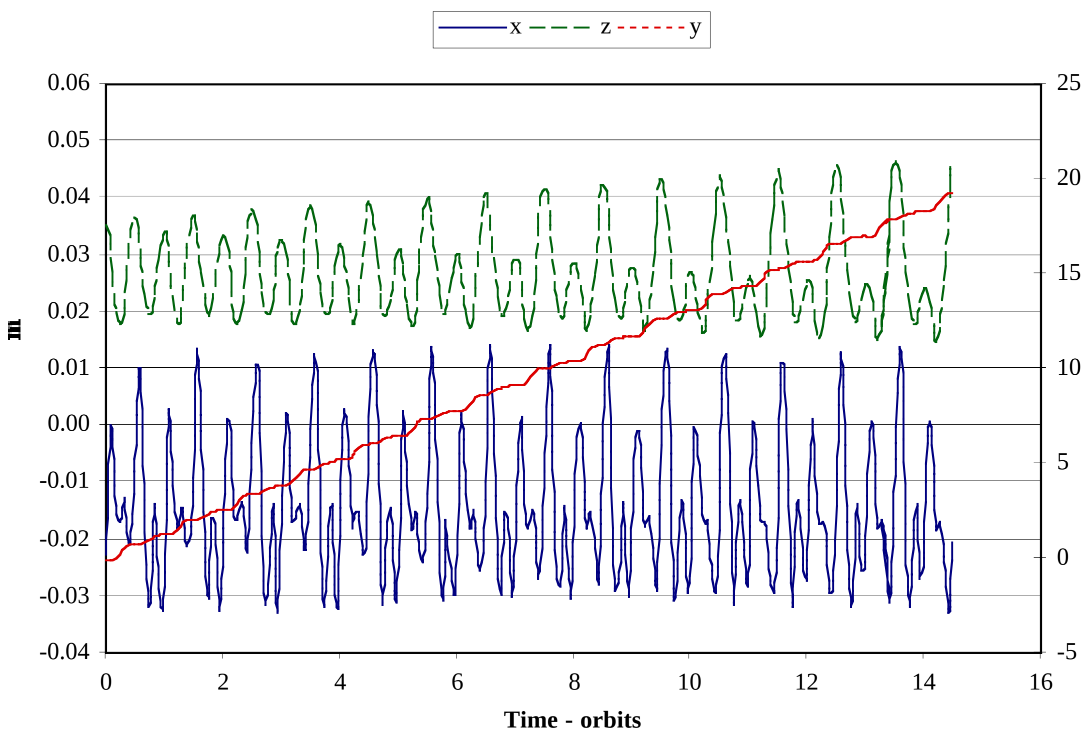


Figure 6 Geometric Method Errors With Gravitational Perturbations

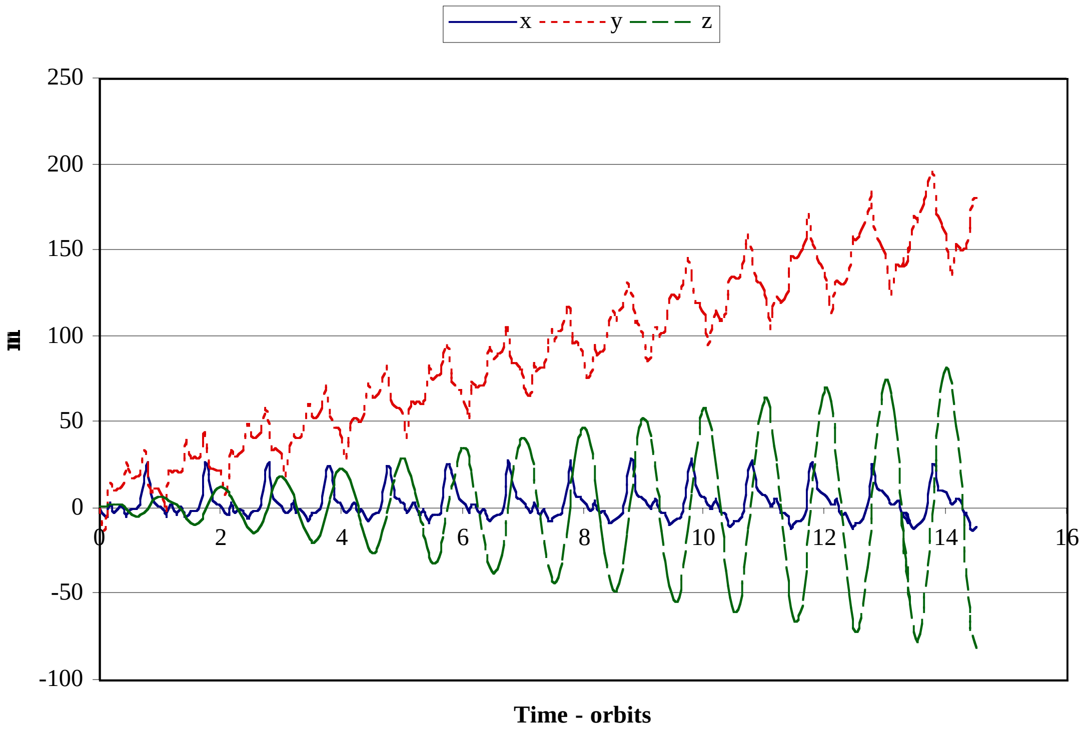


Figure 7 Hill's Equations Errors With Gravitational Perturbations

Hill's equations the mean semi-major axis was used to compute the mean motion, otherwise, the errors would be much larger.

CONCLUSIONS

An algorithm for relating the orbital element changes to the relative motion variables has been developed. This algorithm is used in the development of a state transition matrix that includes the effects of the chief satellite orbit eccentricity and the gravitational perturbations. This state transition matrix was developed by considering the geometry of the problem, not by solving the differential equations. The errors in estimating the relative motion are much less than with using Hill's equations.

Evaluation of the method is continuing. Research is also underway for solving the relative equations of motion using Hamiltonian mechanics and Lie Series.

REFERENCES

1. Folta, D., Newman, L. and Gardner, T., "Foundations of Formation Flying for Mission to Planet Earth and the New Millennium," *AIAA/AAS Astrodynamics Specialists Conference*, July 1996.
2. How, J., Twiggs, R. Weidow, D., Hartman, K. and Bauer, F., "Orion: A Low Cost Demonstration Flying in Space Using GPS," *AIAA/AAS Astrodynamics Specialists Conference*, Boston, MA, August 1998,
3. Campbell, M., Fullmer, R. and Hall C., "The ION-F Formation Flying Experiments", *AAS/AIAA Space Flight Mechanics Conference*, Clearwater, FL, January 23-26, 2000, Paper No. AAS 00-108.
4. Kapila, V., Sparks, A., Buffington, J.M., and Yan, Q.: Spacecraft Formation Flying: Dynamics and Control," *Proceedings of the American Control Conference*, San Diego, CA, June 1999.
5. Kong, E.M., Miller, D.W. and Sedgwick, R.J., "Optimal Trajectories and Orbit Design for Separated Spacecraft Interferometry," TR SERC #13-98, MIT, November 1998.
6. Inalham, G., Busse, F.D. and How, J.P., "Precise Formation Flying Control of Multiple Spacecraft Using Differential Carrier-Phase Differential GPS," *AAS/AIAA Space Flight Mechanics Conference*, Clearwater, FL, January 23-26, 2000, Paper No. AAS 00-109.
7. Schaub, H, and Alfriend, K.T., "J_2 Invariant Reference Orbits for Spacecraft Formations," *Flight Mechanics Symposium*, NASA Goddard Space Flight Center, May 18-20, 1999, Paper No. 11.
8. Schaub, H., Vadali, S.R. and Alfriend, K.T., "Spacecraft Formation Fling Using Mean Orbit Elements," *AAS/AIAA Astrodynamics Specialists Conference,* Girdwood, Alaska, Aug. 16-19, 1999, Paper No. AAS 99-310.
9. Vadali, S.R., Schaub, H. and Alfriend, K.T., "Initial Conditions and Fuel-Optimal Control for Formation Flying of Satellites," *AIAA GN&C Conference*, Portland, OR, Aug. 9-12,1999, Paper No. AIAA 99-4265.
10. Melton, R.G., " Time-Explicit representation of relative Motion Between Elliptical Orbits," ," *AAS/AIAA Astrodynamics Specialists Conference*, Sun valley, ID, Aug. 1997.
11. Brouwer, D., "Solution of the Problem of Artificial Satellite Theory," *The Astronautical Journal*, Vol. 64, No. 1274, 1959, pp. 378-397.

Appendix A - The A Matrix

The non-zero elements of A and A^{-1} are

$$\begin{aligned}
A_{11} &= \frac{R_c}{a_c} \\
A_{12} &= V_r / \dot{\theta}_c \\
A_{14} &= -\frac{R_c}{p_c}\left(2a_c q_{1c} + R_c \cos\theta_c\right) \\
A_{15} &= -\frac{R_c}{p_c}\left(2a_c q_{2c} + R_c \sin\theta_c\right)
\end{aligned} \tag{A1}$$

$$\begin{aligned}
A_{21} &= -05V_{rc}a_c \\
A_{22} &= \sqrt{\frac{\mu}{p_c}}\left(\frac{p_c}{R_c} - 1\right) \\
A_{24} &= \frac{V_{rc}a_c q_{1c}}{p_c} + \sqrt{\frac{\mu}{p_c}} \sin\theta_c \\
A_{25} &= \frac{V_{rc}a_c q_{2c}}{p_c} - \sqrt{\frac{\mu}{p_c}} \cos\theta_c
\end{aligned} \tag{A2}$$

$$\begin{aligned}
A_{32} &= R_c \\
A_{36} &= R_c \cos i_c
\end{aligned} \tag{A3}$$

$$\begin{aligned}
A_{41} &= -1.5V_{tc} / a_c \\
A_{42} &= -V_{rc} \\
A_{44} &= 2\sqrt{\frac{\mu}{p_c}} \cos\theta_c + \frac{3V_{tc}a_c q_{1c}}{p_c} \\
A_{45} &= 2\sqrt{\frac{\mu}{p_c}} \sin\theta_c + \frac{3V_{tc}a_c q_{2c}}{p_c} \\
A_{46} &= V_{rc} \cos i_c
\end{aligned} \tag{A4}$$

$$\begin{aligned}
A_{53} &= R_c \sin\theta_c \\
A_{56} &= -R_c \sin i_c \cos\theta_c \\
A_{63} &= \left(V_{tc}\cos\theta_c + V_{rc}\sin\theta_c\right) \\
A_{66} &= \left(V_{tc}\sin\theta_c - V_{rc}\cos\theta_c\right)\sin i_c
\end{aligned} \tag{A5}$$

$$A_{11}^{-1} = \left(\frac{a_c}{R_c}\right)\left[4 + 6\left(\frac{a_c}{R_c}\right)(q_{1c}\cos\theta_c + q_{2c}\sin\theta_c) + 4\left(\frac{V_{rc}}{V_{tc}}\right)\left(\frac{a_c}{R_c}\right)(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c)\right]$$

$$A_{12}^{-1} = 2\left(\frac{a_c}{R_c}\right)^2 (q_{1c}\sin\theta_c - q_{2c}\cos\theta_c)/\dot{\theta}_c$$

$$A_{13}^{-1} = -2\left(\frac{a_c}{R_c}\right)\left(\frac{V_{rc}}{V_{tc}}\right)\left[1 + 2\left(\frac{a_c}{R_c}\right)(q_{1c}\cos\theta_c + q_{2c}\sin\theta_c) + \left(\frac{V_{rc}}{V_{tc}}\right)\left(\frac{a_c}{R_c}\right)(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c)\right] \tag{A6}$$

$$A_{14}^{-1} = \left(\frac{2a_c}{R_c\dot{\theta}_c}\right)\left[1 + 2\left(\frac{a_c}{R_c}\right)(q_{1c}\cos\theta_c + q_{2c}\sin\theta_c) + \left(\frac{V_{rc}}{V_{tc}}\right)\left(\frac{a_c}{R_c}\right)(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c)\right]$$

$$A_{23}^{-1} = 1/R_c$$

$$A_{25}^{-1} = -A_{65}^{-1}\cos i_c \tag{A7}$$

$$A_{26}^{-1} = -A_{66}^{-1}\cos i_c$$

$$A_{35}^{-1} = \left(\sin\theta_c - (V_{rc}/V_{tc})\cos\theta_c\right)/R_c$$

$$A_{36}^{-1} = \cos\theta_c / V_{tc} \tag{A8}$$

$$A_{41}^{-1} = \sqrt{\frac{p_c}{\mu}}\left[3\cos\theta_c + 2(V_{rc}/V_{tc})\sin\theta_c\right]\dot{\theta}_c$$

$$A_{42}^{-1} = \sqrt{\frac{p_c}{\mu}}\sin\theta_c$$

$$A_{43}^{-1} = -\frac{1}{R_c}\left[\frac{R_c}{P_c}(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c)^2\sin\theta_c + q_{1c}\sin 2\theta_c - q_{2c}\cos 2\theta_c\right] \tag{A9}$$

$$A_{44}^{-1} = \sqrt{\frac{p_c}{\mu}}\left[2\cos\theta_c + (V_{rc}/V_{tc})\sin\theta_c\right]$$

$$A_{45}^{-1} = -q_{2c}\cot i_c\left[\cos\theta_c + (V_{rc}/V_{tc})\sin\theta_c\right]/R_c$$

$$A_{46}^{-1} = q_{2c}\cot i_c\sin\theta_c / V_{tc}$$

$$
\begin{aligned}
A_{51}^{-1} &= \sqrt{\frac{p_c}{\mu}}\left[3\sin\theta_c - 2\left(V_{rc} / V_{tc}\right)\cos\theta_c\right]\dot{\theta}_c \\
A_{52}^{-1} &= -\sqrt{\frac{p_c}{\mu}}\cos\theta_c \\
A_{53}^{-1} &= \frac{1}{R_c}\left[\frac{R_c}{P_c}\left(q_{1c}\sin\theta_c - q_{2c}\cos\theta_c\right)^2\cos\theta_c + q_{2c}\sin 2\theta_c + q_{1c}\cos 2\theta_c\right] \\
A_{54}^{-1} &= \sqrt{\frac{p_c}{\mu}}\left[2\sin\theta_c - \left(V_{rc} / V_{tc}\right)\cos\theta_c\right] \\
A_{55}^{-1} &= q_{1c}\cot i_c\left[\cos\theta_c + \left(V_{rc} / V_{tc}\right)\sin\theta_c\right] / R_c \\
A_{56}^{-1} &= -q_{1c}\cot i_c \sin\theta_c / V_{tc}
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
A_{65}^{-1} &= -\left[\cos\theta_c + \left(V_{rc} / V_{tc}\right)\sin\theta_c\right] / R_c \sin i_c \\
A_{66}^{-1} &= \sin\theta_c / V_{tc}\sin i_c
\end{aligned}
\tag{A12}
$$