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LINEAR DYNAMICS AND STABILITY ANALYSIS OF A COULOMB TETHER FORMATION

**Arun Natarajan*[*] and Hanspeter Schaub*[†]

The linear dynamics and stability analysis of a Coulomb tether formation is investigated. Here the relative distance between two satellites is controlled using electrostatic Coulomb forces. A charge feedback law is introduced to stabilize the relative distance between the satellites to a constant value. The two craft as being connected by an electrostatic tether which is capable of both tensile and compressive forces. As a result, the two-craft formation will essentially act as a long, slender near-rigid body. Inter-spacecraft Coulomb forces cannot influence the inertial angular momentum of this formation. However, the gravity gradient effect can be exploited to stabilize the attitude of this Coulomb tether formation about an orbit nadir direction. The Coulomb tether has been modeled as a massless, elastic component. The elastic strength of this connection is controlled through a spacecraft charge control law.

INTRODUCTION

The concept of formation flying using electrostatic propulsion was introduced in References 1–3. The electrostatic (Coulomb) charge of spacecraft is varied by active emission of either negative electric charges (electrons) or positive electric charges (ions). The resulting changes in inter-spacecraft Coulomb forces are used to control the relative motion of the spacecraft. This novel concept of propellantless relative navigation control has many advantages over conventional thrusters like ion engines. For example, this method of propulsion has been shown to require essentially no consumables, require very little electric power to operate (often less than 1 Watt), and can be controlled with a very high bandwidth (zero to maximum charge transition times are of the order of milli-seconds). Thus, this propulsion concept could enable high precision formation flying with separation distances ranging 10–100 meters. It is also a very clean method of propulsion compared to ion engines, thereby avoiding the thruster plume contamination issue with neighboring satellites. For the suggested range of separation distances, the plume-impingement problem of high-efficiency ion engines would be severe. Proposed uses of the Coulomb propulsion concept include high-accuracy, wide-field-of-view optical interferometry missions at geostationary altitudes,[2] controlling clusters of spacecraft to maintain a bounded shape,[3] as well as the use of drone-worker concept where dedicated craft place a sensor in space using Coulomb forces.[4] This paper introduces a new application of the Coulomb propulsion concept. The electrostatic force is used to control the separation distance between two physically unconnected craft. Due to the similarities with using a tether cable to connect two craft, this concept is called a Coulomb tether formation. Note that contrary to traditional tethers, the Coulomb tether is capable of receiving both tensile and compressive forces. Further, the

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stiffness of the satellite connection can be controlled through feedback control laws. This will allow for the Coulomb tether stiffness to be varied with changing mission requirements.

While the Coulomb propulsion concept has many exciting advantages, it does come at the price of greatly increased coupling and nonlinearity of the charged spacecraft equations of motion. The relative motion of all other neighboring charged craft will be affected by changing the charge of a single craft. Further, with the Coulomb forces being formation-internal forces, some constraints are applied to all feasible charged spacecraft motion. In particular, Coulomb forces cannot be used to change the total inertial formation angular momentum vector.[5,6] As a result, these spacecraft charges cannot be used to reorient a formation as a whole to a new orientation. An external influence must be used or generated through thrusters to reorient a Coulomb formation. With the Coulomb tether formation, we seek to exploit the gravity gradient torque effect that rigid bodies experience in orbit. Spacecraft do not experience the same gravitational pull on all parts of their body. The sections which are closer to the Earth are attracted more strongly than those that are further away. This force or gravity gradient[7] has been used in stabilizing some satellites. To guarantee linear stability of rigid body attitudes in orbit, the principal inertias of the body must satisfy well-known constraints. Typically gravity-gradient stabilized satellites are tall and slender, and aligned with the local nadir direction. The same concept of stabilization can be extended to the two spacecraft Coulomb tether concept where the craft are assumed to be flying apart by a few dozen meters. By employing a charge feedback law to stabilize the spacecraft separation distance (making the formation act as a rigid, slender rod), the gravity gradient torque will assist in stabilizing the formation attitude.

The study of electrostatic charging data of SCATHA spacecraft[8] in GEO has shown that the spacecraft can naturally charge to very high voltages in low plasma environments such as at GEO. The level of natural charge depends on the current solar activity. Further, this mission demonstrated that the spacecraft charge could be actively controlled. The Coulomb propulsion has its own set of limitations, however. The magnitude of Coulomb electrostatic force is inversely proportional to the square of separation distance, which makes this method effective only for close formations of the order of 10-100 m, depending on the maximum allowable level of spacecraft charge. Moreover, if charged plasma particles are present in the space, the effectiveness of Coulomb force is diminished with the electric field dropping off exponentially. The severity of this drop is measured using the Debye length.[9,10] For low Earth orbits (LEO), the Debye length is of the order of centimeters, making the Coulomb formation flying concept impractical. At geostationary orbits (GEO) or higher, where the plasma environment is milder, the Debye length is about 100-1400 m. The Coulomb formation flying concepts can be comfortably applied at this altitude. King et. al.[1] found analytical solutions for Hill-frame invariant Coulomb formations. Here spacecraft are placed at specific location in the rotating Hill frame with specific electrostatic charges. As a result the Coulomb forces perfectly cancel all natural orbital accelerations, causing the satellites to remain fixed or static as seen by the Hill frame. However, in this study the charge was held constant in their analysis. The discovered static Coulomb formations were all found to be unstable.

Thus, this paper attempts to exploit the known stability characteristics of orbital rigid body motion under a gravity gradient field and examine its applicability to a Coulomb tethered two-spacecraft system. To avoid the very small plasma Debye lengths found at

LEO, the Coulomb tether formation studied will be at GEO. The formation center of mass or chief motion is assumed to be circular. The paper is organized as follows. After discussing the charged spacecraft equations of motion, the equations are rewritten using spherical coordinates and linearized for small departure angles. A feedback charge control law is introduced to stabilize the separation distance, followed by a combined attitude and separation distance linear stability analysis. A numerical simulation illustrates the results.

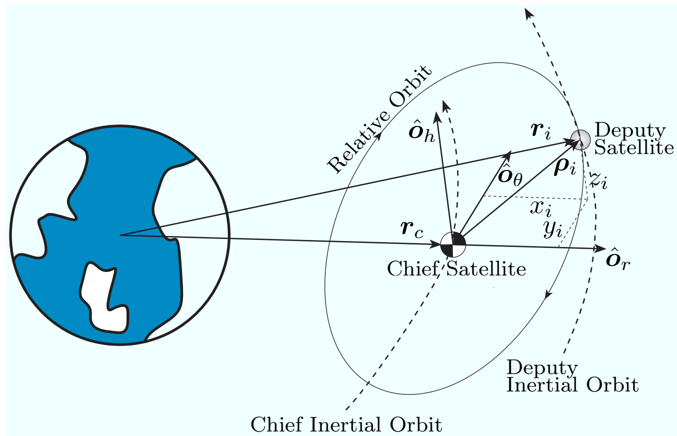


Figure 1: Rotating Hill Coordinate System Used to Describe the Relative Position of the Satellites

STATIC (RIGID) FORMATION DYNAMICS

Let us briefly review the equations of motion of a cluster of charged spacecraft. The Clohessy-Wiltshire-Hill's equations[11,12] are commonly used for formation studies. These equations express the linearized motion of one satellite relative to a circularly orbiting reference point or chief location. Note that this chief location does not have to be actually occupied by a satellite. For the present discussion, the formation chief location is set to be equal to the formation center of mass. The various satellites in a formation are called the deputy satellites. The system of Cartesian coordinates used to describe the relative motion of a satellite with respect to the chief location is defined in the rotating Hill orbit frame $\mathcal{O} : \{\hat{\boldsymbol{o}}_r, \hat{\boldsymbol{o}}_\theta, \hat{\boldsymbol{o}}_h\}$ as shown in Figure 1. The origin of the coordinate system is chosen to be the formation center of mass or chief location. The Cartesian x, y and z coordinates are the vector components of the relative position vector

$$\boldsymbol{\rho} = {}^{\mathcal{O}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

along the orbit nadir, the orbital velocity vector and the normal to the orbit plane respectively. Assuming that the Coulomb formation contains N satellites. The CW equations of

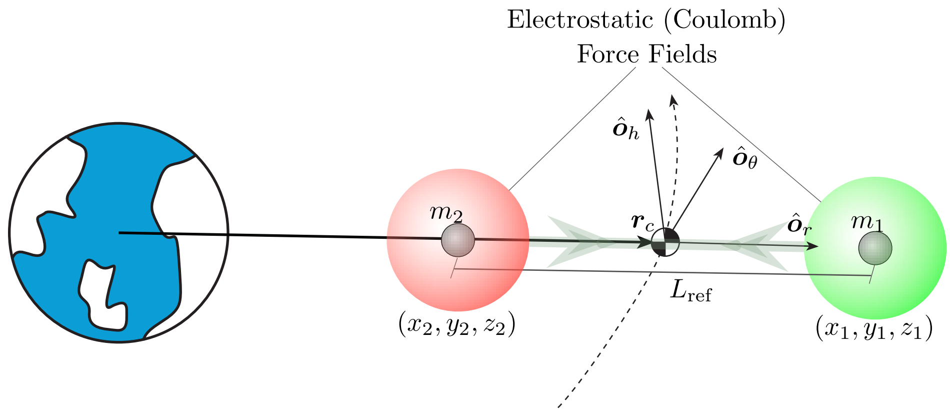


Figure 2: Coulomb Tethered Two Satellite Formation with the Satellites Aligned Along the Orbit Nadir Direction

the i^{th} deputy with respect to the chief are expressed as

$$\ddot{x}_i - 2\Omega\dot{y}_i - 3\Omega^2 x_i = \frac{k_c}{m_i}\sum_{j=0}^{n}\frac{(x_i - x_j)}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|^3} q_i q_j e^{-\rho_{ij}/\lambda_d} \quad j \neq i \tag{2a}$$

$$\ddot{y}_i + 2\Omega\dot{x}_i = \frac{k_c}{m_i}\sum_{j=0}^{n}\frac{(y_i - y_j)}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|^3} q_i q_j e^{-\rho_{ij}/\lambda_d} \quad j \neq i \tag{2b}$$

$$\ddot{z}_i + \Omega^2 z_i = \frac{k_c}{m_i}\sum_{j=0}^{n}\frac{(z_i - z_j)}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|^3} q_i q_j e^{-\rho_{ij}/\lambda_d} \quad j \neq i \tag{2c}$$

where $\boldsymbol{\rho}_i = (x_i, y_i, z_i)^T$ gives the position vector of the i^{th} satellite in Hill frame components, m_i is the satellite mass, q_i the satellite charge. The chief position vector $\boldsymbol{r}_c$ is assumed to have a constant orbital rate of $\boldsymbol{\Omega} = \sqrt{\mu/r_c^3}$. The parameter $k_c = 8.99 \cdot 10^9 \text{Nm}^2/\text{C}^2$ is the Coulomb's constant, while the parameter λ_d is the Debye length. Because the Coulomb tether formations are assumed to be at GEO where the Debye length is much larger than the typical Coulomb tether length, the Debye length influence is ignored as a higher order term for the remainder of the paper. Note that these relative equations of motion of a charged spacecraft contain linearized orbital dynamics, while retaining the full nonlinear Coulomb force expression. In fact, it is this very nonlinear Coulomb force term that causes the strong and complex coupling between the spacecraft motions.

The formation geometry of the ideal 2-craft Coulomb tether formation is shown in Figure 2. As will be shown later in this section, there exists a 2-craft static Coulomb formation solution where both masses must be aligned equal distances away from the chief along the nadir direction. The ideal separation distance is called L_{ref}. If each craft has a certain charge, then the resulting Coulomb forces will perfectly cancel the linearized orbital accelerations in the Hill frame. As a result, the two craft would each remain aligned in the chief nadir direction and perform non-Keplerian motions. To an external observer the two physically unconnected craft would appear to both be performing perfectly circular motions, but with a non-Keplerian orbit period for their individual altitudes. The invisible Coulomb tether is applied the required inter-spacecraft force, similar to how a cable tether could provide the required tension between the craft to maintain such non-Keplerian orbits.

Because the Coulomb tether formation considered only contains two spacecraft, the CW equations in Eq. (2) for satellite 1 can be simplified to

$$\ddot{x}_1 - 2\Omega\dot{y}_1 - 3\Omega^2 x_1 = \frac{k_c}{m_1}\frac{(x_1 - x_2)}{L^3}q_1 q_2 \tag{3a}$$

$$\ddot{y}_1 + 2\Omega\dot{x}_1 = \frac{k_c}{m_1}\frac{(y_1 - y_2)}{L^3}q_1 q_2 \tag{3b}$$

$$\ddot{z}_1 + \Omega^2 z_1 = \frac{k_c}{m_1}\frac{(z_1 - z_2)}{L^3}q_1 q_2 \tag{3c}$$

Because the Hill frame $\mathcal{O}$ origin is assumed to be identical to the formation center of mass, the center of mass constraint dictates that[5,6]

$$m_1\boldsymbol{\rho}_1 + m_2\boldsymbol{\rho}_2 = 0 \tag{4}$$

Thus, by controlling the satellite 1 motion, through the center of mass constraint the motion of the second satellite is also determined.

In order for this top-down spacecraft formation to remain statically fixed relative to the orbit frame $\mathcal{O}$, the CW equations in Eq. (3) must be satisfied with zero initial velocity and acceleration for each vehicle

$$\dot{x}_i = \ddot{x}_i = \dot{y}_i = \ddot{y}_i = \dot{z}_i = \ddot{z}_i = 0$$

For a two-craft Coulomb formation, this is only possible if the relative positions are given through:

$$m_1 x_1 + m_2 x_2 = 0 \tag{5a}$$

$$x_1 - x_2 = L \tag{5b}$$

$$x_1 = \frac{m_2}{m_1 + m_2}L \tag{5c}$$

$$x_2 = -\frac{m_1}{m_1 + m_2}L \tag{5d}$$

$$y_1 = y_2 = z_1 = z_2 = 0 \tag{5e}$$

Substituting the above conditions and constraints in Eq. (3), we arrive at the following two spacecraft charge conditions.

$$\frac{k_c}{m_1}\frac{q_1 q_2}{L^2} + 3\Omega^2\frac{m_2 L}{m_1 + m_2} = 0 \quad \Rightarrow \quad q_1 q_2 = -3\Omega^2\frac{L^3}{k_c}\frac{m_1 m_2}{m_1 + m_2} \tag{6a}$$

$$\frac{k_c}{m_2}\frac{q_1 q_2}{L^2} + 3\Omega^2\frac{m_1 L}{m_1 + m_2} = 0 \quad \Rightarrow \quad q_1 q_2 = -3\Omega^2\frac{L^3}{k_c}\frac{m_1 m_2}{m_1 + m_2} \tag{6b}$$

Thus, the ideal product of charges Q_{ref} needed to achieve this static Coulomb formation is

$$Q_{\text{ref}} = q_1 q_2 = -3\Omega^2\frac{L^3}{k_c}\frac{m_1 m_2}{m_1 + m_2} \tag{7}$$

Thus, if the satellites are placed at the locations shown in Eq. (5), and have the charges q_1 and q_2 satisfy Eq. (7), then the satellites will appear to be frozen or fixed as seen by the rotating frame $\mathcal{O}$. Note that this reference charge product term will be negative! This dictates that the spacecraft charges q_1 and q_2 will have opposite charge signs.

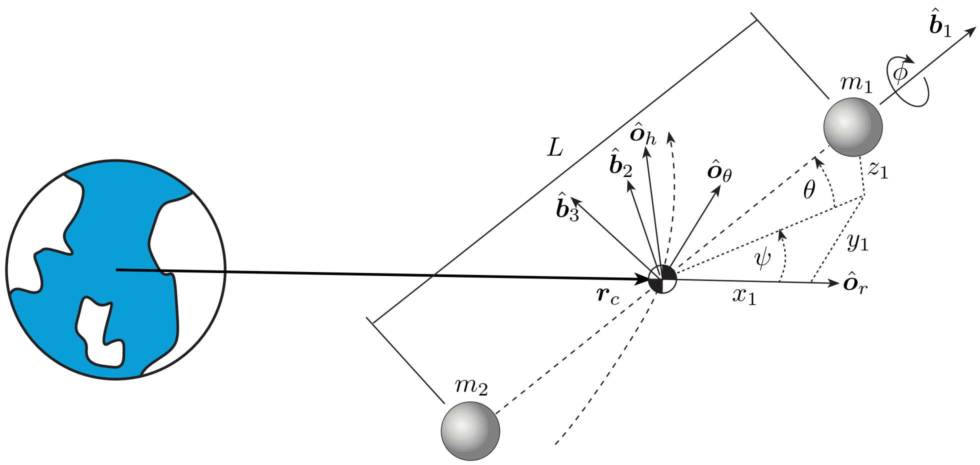


Figure 3: Euler Angles Representing the Attitude of Coulomb Tether with Respect to the Orbit Frame

LINEARIZED ORBITAL PERTURBATION

The constant charge computed in accordance with Eq. (7) is adequate to maintain the satellite formation if there is no perturbation of the orbit. In the event of a perturbation, the relative separation will become unstable and the satellites will separate. In this section, we establish a relationship between these position and charge states by considering small perturbations about the established reference states.

Let us treat the two-craft formation as if it were a solid, physically connected body. Thus, we introduce the body-fixed coordinate frame $\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ where $\hat{\boldsymbol{b}}_1$ is aligned with the relative position vector $\boldsymbol{\rho}_1$. Note that if the body is at the ideal Coulomb tether orientation where the masses are aligned equally along the orbit nadir direction $\hat{\boldsymbol{o}}_r$, then we find that the $\mathcal{O}$ and $\mathcal{B}$ frame orientation vectors are identical. The relative position vector of mass m_1 in body fixed axes is given by

$$\boldsymbol{\rho}_1 = \frac{m_2}{m_1 + m_2} L\hat{\boldsymbol{b}}_1 + 0\hat{\boldsymbol{b}}_2 + 0\hat{\boldsymbol{b}}_3 \tag{8}$$

Let the 3-2-1 Euler angles (ψ, θ, ϕ) represent the Coulomb tether $\mathcal{B}$ frame attitude relative to the orbit frame $\mathcal{O}$ for small angular perturbations as shown in Figure 3. Because point masses are being considered, the rotation about $\hat{\boldsymbol{b}}_1$ (angle ϕ) can be neglected. The direction cosine matrix $[BO(\psi, \theta)]$, which relates the $\mathcal{O}$ frame to $\mathcal{B}$ frame, is given by

$$[BO] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \tag{9}$$

Using small angle approximations for the trigonometric functions, the position vector of mass m_1 in $\mathcal{O}$ frame can be written as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = [BO]^T \begin{pmatrix} \frac{m_2}{m_1+m_2}L \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} \frac{m_2}{m_1+m_2}L \\ \psi\frac{m_2}{m_1+m_2}L \\ -\theta\frac{m_2}{m_1+m_2}L \end{pmatrix} \tag{10}$$

Taking the derivative of this expression, the linearized Hill frame relative velocity coordinates are found to be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} \approx \frac{m_2}{m_1+m_2} \begin{pmatrix} \dot{L} \\ \psi\dot{L} + \dot{\psi}L \\ -\theta\dot{L} - \dot{\theta}L \end{pmatrix} \tag{11}$$

The distance L between the two masses m_1 and m_2 is given by

$$L^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{12}$$

Using the center of mass condition in Eq. (4), this can be simplified to

$$L^2 = \left(\frac{m_1+m_2}{m_2}\right)^2 (x_1^2 + y_1^2 + z_1^2) \tag{13}$$

Differentiating the above expression twice and substituting for the second derivatives from Eq. (3), we obtain

$$\dot{L}^2 + L\ddot{L} = \left(\frac{m_1+m_2}{m_2}\right)^2 \Big(\dot{x}_1^2 + x_1\Big(2\Omega\dot{y}_1 + 3\Omega^2 x_1 + \frac{k_c}{m_1}\frac{(x_1-x_2)}{L^3}Q\Big) + \dot{y}_1^2 + y_1\Big(-2\Omega\dot{x}_1 + \frac{k_c}{m_1}\frac{(y_1-y_2)}{L^3}Q\Big) + \dot{z}_1^2 + z_1\Big(-\Omega^2 z_1 + \frac{k_c}{m_1}\frac{(z_1-z_2)}{L^3}Q\Big)\Big) \tag{14}$$

Transforming the Cartesian coordinates (x_1, y_1, z_1) to spherical coordinates (L, ψ, θ) using Eq. (10) & Eq. (11) and neglecting higher order terms in ψ and θ, we get the linearized differential equation of the separation distance L.

$$\ddot{L} = (2\Omega\dot{\psi} + 3\Omega^2)L + \frac{k_c}{m_1}Q\frac{1}{L^2}\frac{m_1+m_2}{m_2} \tag{15}$$

Note the following special case. Assume that the charge product term Q is zero (i.e. classical Keplerian motion), and that the satellites are initial at rest with $\dot{\psi} = 0$. In this case the separation distance equations of motion simplify to

$$\ddot{L} - 3\Omega^2 L = 0$$

This unstable oscillator equation demonstrates that without any Coulomb force active, this formation could not remain at the specific locations.

Next the separation distance equations of motion are linearized about small variations in length δL and small variations in the product charge term δQ. The reference separation length L_{ref} is determined by the mission requirement. The reference charge product term is determined through the L_{ref} choice and the constraint in Eq. (7).

$$L = L_{\text{ref}} + \delta L \tag{16a}$$
$$Q = Q_{\text{ref}} + \delta Q \tag{16b}$$

Substituting these L and Q definitions into Eq. (15) and linearizing leads to

$$\delta\ddot{L} = (2\Omega L_{\text{ref}})\dot{\psi} + (9\Omega^2)\delta L + \left(\frac{k_c}{m_1}\frac{1}{L_{\text{ref}}^2}\frac{m_1+m_2}{m_2}\right)\delta Q \tag{17}$$

This equation establishes the desired relationship between the additional charge product δQ required and the change in relative separation of the satellites. We observe that this relation is coupled to the body frame yaw rate $\dot{\psi}$. In order to obtain an expression for this, we resort to a stability analysis using gravity gradient.

To develop a feedback law to control the separation distance using the Coulomb forces, we first treat the small charge product variation δQ as a control variable. Because the charge of each craft causes a force on craft 1 along the relative position vector, the Coulomb charges can be used to control the spacecraft separation distance. By defining

$$\delta Q = \frac{m_1 m_2 L_{\text{ref}}^2}{(m_1+m_2)\,k_c}(-C_1\delta L - C_2\delta\dot{L}) \tag{18}$$

the closed-loop separation distance dynamics become

$$\delta\ddot{L} + (C_1 - 9\Omega^2)\delta L + C_2\delta\dot{L} - (2\Omega L_{\text{ref}})\dot{\psi} = 0 \tag{19}$$

This control law provides both proportional and derivative feedback of δL. Because the δL differential equation does not contain a damping term $\delta\dot{L}$, the inclusion of the derivative feedback is essential to ensure asymptotic convergence. Note that in the absence of the yaw rate term $\dot{\psi}$, these closed-loop dynamics would be stable if $C_1 > 9\Omega^2$ and $C_2 > 0$. However, due to the coupling with the yaw (in-orbit-plane) rotation, the complete Coulomb tether motion must be analyzed for stability.

To implement this charge feedback control law, the spacecraft charges q_1 and q_2 must be determined. The value of Q_{ref} is determined through Eq. (7), while the value of δQ is given by the feedback law expression in Eq. (18). Thus, the spacecraft charges q_1 and q_2 must satisfy

$$q_1 q_2 = Q_{\text{ref}} + \delta Q \tag{20}$$

There is an infinite number of solutions to the above constraint. To keep the charges equal in magnitude across the craft, the following implementation was used.

$$q_1 = \sqrt{|Q_{\text{ref}} + \delta Q|} \tag{21}$$
$$q_2 = -q1 \tag{22}$$

Note that here $Q_{\text{ref}} + \delta Q < 0$ because $\delta Q \ll Q_{\text{ref}}$ and $Q_{\text{ref}} < 0$. With this charging convention $q_1 > 0$ and $q_2 < 0$.

STABILITY ANALYSIS USING GRAVITY GRADIENT

In this section we analyze the stability of both the Coulomb tether attitude (ψ, θ) and the separation distance L. The gravity gradient torque is included to exert an external torque onto the Coulomb tether. Let the orbit angular velocity vector relative to the inertial frame $\mathcal{N}$ be given by

$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = \boldsymbol{\omega} = \Omega \hat{o}_h \tag{23}$$

Euler's rotational equation of motion with time varying inertia matrix $[I]$ and gravity gradient torque vector $\boldsymbol{L}_G$ is given in body frame $\mathcal{B}$ by

$${}^{\mathcal{B}}[I]\,{}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}[\dot{I}]\,{}^{\mathcal{B}}\boldsymbol{\omega} + {}^{\mathcal{B}}[\tilde{\boldsymbol{\omega}}]\,{}^{\mathcal{B}}[I]\,{}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{L}_G \tag{24}$$

where the notation $[\tilde{\boldsymbol{\omega}}]\boldsymbol{x} \equiv \boldsymbol{\omega} \times \boldsymbol{x}$ is used. Using the direction cosine matrix definition in Eq. (10), the orbit angular velocity vector can be written as

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = [BO]\,{}^{\mathcal{O}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = \begin{bmatrix} -\Omega\sin\theta \\ 0 \\ \Omega\cos\theta \end{bmatrix} \tag{25}$$

The yaw and pitch rates of the Coulomb tether body frame $\mathcal{B}$ relative to the orbit $\mathcal{O}$ frame yield

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \begin{bmatrix} -\sin\theta & 0 \\ 0 & 1 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \end{bmatrix} \tag{26}$$

The Coulomb tether body frame angular velocity vector relative to the inertial frame $\mathcal{N}$ is

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = {}^{\mathcal{B}}\begin{bmatrix} -\sin\theta\dot{\psi} - \Omega\sin\theta \\ \dot{\theta} \\ \cos\theta\dot{\psi} + \Omega\cos\theta \end{bmatrix} \tag{27}$$

Linearizing the Eq. (27) about small yaw and pitch angles, we get

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \approx {}^{\mathcal{B}}\begin{bmatrix} -\Omega\theta \\ \dot{\theta} \\ \dot{\psi} + \Omega \end{bmatrix} \tag{28}$$

Taking the inertial derivative of this vector and noting that Ω is constant in this application, the $\mathcal{B}$ frame angular acceleration is

$${}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \approx {}^{\mathcal{B}}\begin{bmatrix} -\Omega\dot{\theta} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} \tag{29}$$

The moment of inertia matrix is expressed as[7]

$$[I] = -m_1[\tilde{\boldsymbol{\rho}}_1][\tilde{\boldsymbol{\rho}}_1] - m_1[\tilde{\boldsymbol{\rho}}_2][\tilde{\boldsymbol{\rho}}_2] \tag{30}$$

For the 2-craft Coulomb tether formation, using the center of mass definition in Eq. (4), the inertia matrix is trivially given in the body frame $\mathcal{B}$ as

$$^{\mathcal{B}}[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{31}$$

where

$$I = \frac{m_1 m_2}{m_1 + m_2} L^2 \tag{32}$$

The $\mathcal{B}$-frame derivative of the inertia matrix is

$$^{\mathcal{B}}[\dot{I}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dot{I} & 0 \\ 0 & 0 & \dot{I} \end{bmatrix} \tag{33}$$

where

$$\dot{I} = 2\frac{m_1 m_2}{m_1 + m_2} L\dot{L} = 2\frac{m_1 m_2}{m_1 + m_2}(L_{\text{ref}} + \delta L)\delta\dot{L} \tag{34}$$

because L_{ref} =constant.

The center of mass position vector $\boldsymbol{R}_c$, given in $\mathcal{O}$ frame components as

$$^{\mathcal{O}}\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_c \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

is transformed to the $\mathcal{B}$ frame as

$$^{\mathcal{B}}\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_{c1} \\ R_{c2} \\ R_{c3} \end{pmatrix} = {}^{\mathcal{B}}\begin{pmatrix} \cos\theta\cos\psi \\ -\sin\psi \\ \sin\theta\cos\psi \end{pmatrix} R_c \tag{36}$$

Reference 7 provides the following expression for gravity gradient:

$$^{\mathcal{B}}\begin{bmatrix} L_{G_1} \\ L_{G_2} \\ L_{G_3} \end{bmatrix} = \frac{3GM_e}{R_c^5}\begin{bmatrix} R_{c2}R_{c3}(I_{33} - I_{22}) \\ R_{c1}R_{c3}(I_{11} - I_{33}) \\ R_{c1}R_{c2}(I_{22} - I_{11}) \end{bmatrix} \tag{37}$$

After substituting for R_{c_i} from Eq. (36) and using the known value of Ω from Kepler's equation, namely,

$$\frac{3GM_e}{R_c^3} = \Omega^2 \tag{38}$$

the gravity gradient torque vector acting on the Coulomb tether body frame is written as

$$^{\mathcal{B}}\boldsymbol{L}_G \cong 3\Omega^2 \begin{bmatrix} 0 \\ -I\theta \\ -I\psi \end{bmatrix} \tag{39}$$

Next, we are able to substitute these results for $\boldsymbol{L}_G$, ${}^{\mathcal{B}}[\dot{I}]$, ${}^{\mathcal{B}}[I]$, $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ back into Euler's rotational equations of motion in Eq. (24). After simplifying the algebra, the resulting linearized attitude dynamics of the Coulomb tether body frame $\mathcal{B}$ are written with the separation distance differential equation as:

$$\ddot{\theta} + 4\Omega^2\theta = 0 \tag{40a}$$

$$\ddot{\psi} + \frac{2\Omega}{L_{\text{ref}}}\delta\dot{L} + 3\Omega^2\psi = 0 \tag{40b}$$

$$\delta\ddot{L} + C_2\delta\dot{L} - (2\Omega L_{\text{ref}})\dot{\psi} + (C_1 - 9\Omega^2)\delta L = 0 \tag{40c}$$

Thus, Eqs. (40a) – (40c) are the linearized equations of motion of the Coulomb tether body. It can be observed from these equations that θ is decoupled and its equation is that of a simple oscillator. This decoupling is analogous to what occurs with the linearized rigid body attitude dynamics subject to a gravity gradient torque. Because the θ motion not coupled to the tether charge product term δQ, or the separation distance variation δL, it is not possible to control the pitch motion θ with the Coulomb charge. The yaw motion $\psi(t)$ is coupled with the $\delta L(t)$ motion in the form of a driving force which may make it amenable to asymptotic stabilization by controlling the charge.

The values of gain C_1 and C_2 can be tuned to meet the stability requirements using Routh-Hurwitz stability criterion. The characteristic equation for the coupled δL and ψ equations was found to be

$$\lambda^4 + C_2\lambda^3 + (C_1 - 2\Omega^2)\lambda^2 + 3C_2\Omega^2\lambda + 3(C_1\Omega^2 - 9\Omega^4) = 0 \tag{41}$$

While the linearized closed-loop dynamics do depend on the Coulomb tether reference length L_{ref}, note that the characteristic equation does not. The roots of Eq. (41) do depend on the mean orbit rate Ω. To ensure asymptotic stability, roots of this equation should have negative real part. The constraints on the gains C_1 and C_2 for meeting this condition were identified by constructing a Routh table and were found to be

$$C_1 > 9\Omega^2 \tag{42a}$$

$$C_2 > 0 \tag{42b}$$

Incidentally, these constraints also ensure the stability of δL equation ignoring the $\dot{\psi}$ term. We fix the gain C_2 so that the δL equation, with the $\dot{\psi}$ term neglected, is critically damped. Let n be a positive, real feedback gain scaling factor. Then we can define

$$C_1 = n\Omega^2 \tag{43}$$

Further, the critical damping requires that

$$C_2 = 2\Omega\sqrt{n-9} \tag{44}$$

The natural frequency of the ψ equation is $\sqrt{3}\Omega$ and is not affected by the choice of C_1 and C_2. Whereas the natural frequency for δL equation is $\sqrt{(n-9)}\Omega$. The value of $n = 12$ will match these frequencies making the $\dot{\psi}$ coupling term in δL equation to contribute to the damping. A similar remark applies to the ψ equation. Therefore, this choice should be optimal. To further elucidate this aspect, the real part of the roots of the characteristic equation was examined numerically. Since the roots are complex conjugate pairs, there are only two real parts that need to be considered. A plot of the two real parts for n varying between 9 and 40 are shown in Figure 4. It is seen that the real part of one of the roots will tend to zero on either side of $n = 12$. Hence, value of n very close to 9 or those which are large will tend to slow down the asymptotic convergence.

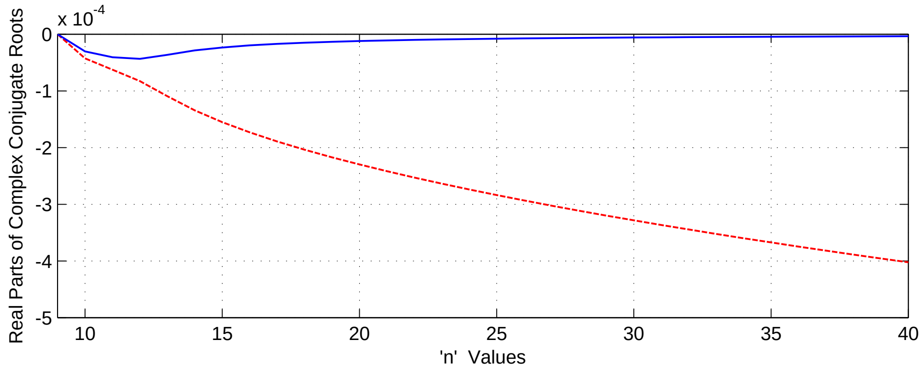


Figure 4: Real Part of the Roots of the Characteristic Equation for Varying Gains

NUMERICAL SIMULATION

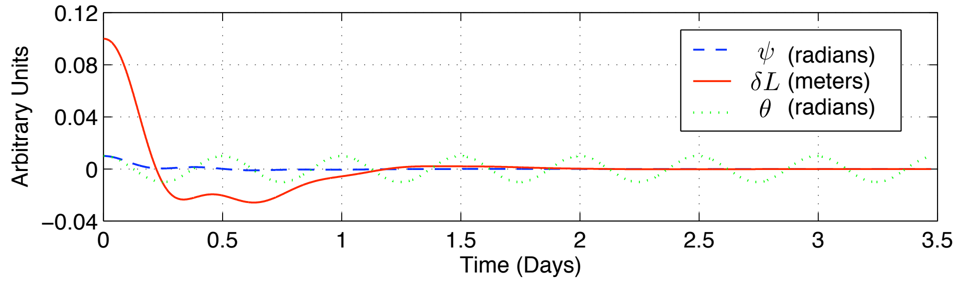
A numerical simulation is presented to illustrate the performance and stability of a Coulomb tether formation. The simulation parameters that were used are listed in Table 1. The initial attitude values are set to $\psi = 0.01$ radians and $\theta = 0.01$ rad. The separation length error (Coulomb tether length error) is $\delta L = 0.1$ meters. All initial rates are set to zero through $\dot{\psi} = \delta\dot{L} = \dot{\theta} = 0$.

The choice of values for the gains C_1 and C_2 should not only satisfy the stability criterion mentioned in Eq. (42) but also should be such as to lead to critical damping. To estimate such gains, the $\dot{\psi}$ term in δL equation is ignored (Eq. (17)), and the approximate values of C_1 and C_2 are found to be $10\Omega^2$ and 2Ω respectively. The equations were solved using a Runge-Kutta integrator with a time step of 100 seconds, which is sufficiently smaller than the period of natural oscillation.

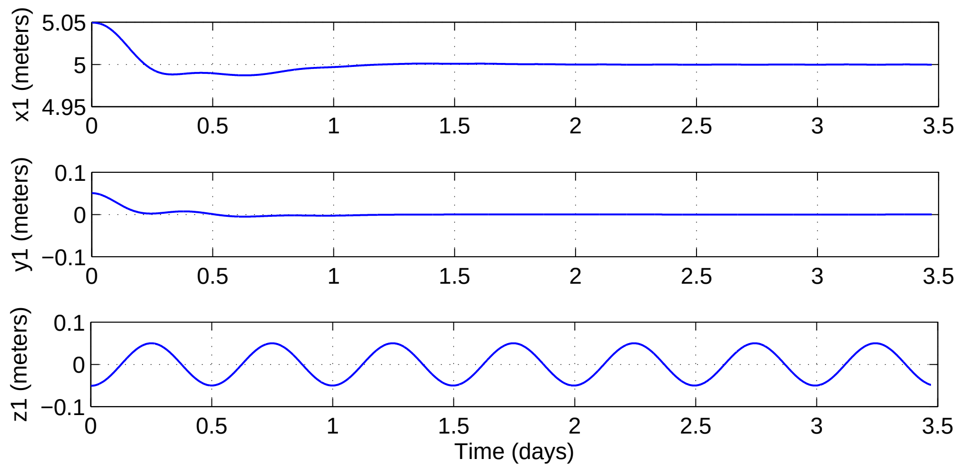
Table 1: Input Parameters Used in Simulation

Parameter	Value	[Units]
m_1	150	kg
m_2	150	kg
L_{ref}	10	m
k_c	8.99×10^9	$\frac{\text{Nm}^2}{\text{C}^2}$
Q_{ref}	-1.3306×10^{-1}	μC^2
Ω	7.2915×10^{-5}	rad/sec

Figure 5(a) shows the Coulomb tether motion if the linearized spherical coordinates



(a) Spherical Coordinates



(b) Hill Frame Cartesian Coordinates

Figure 5: Simulation Results of Integrating the Linearized Spherical Coordinates Differential Equations.

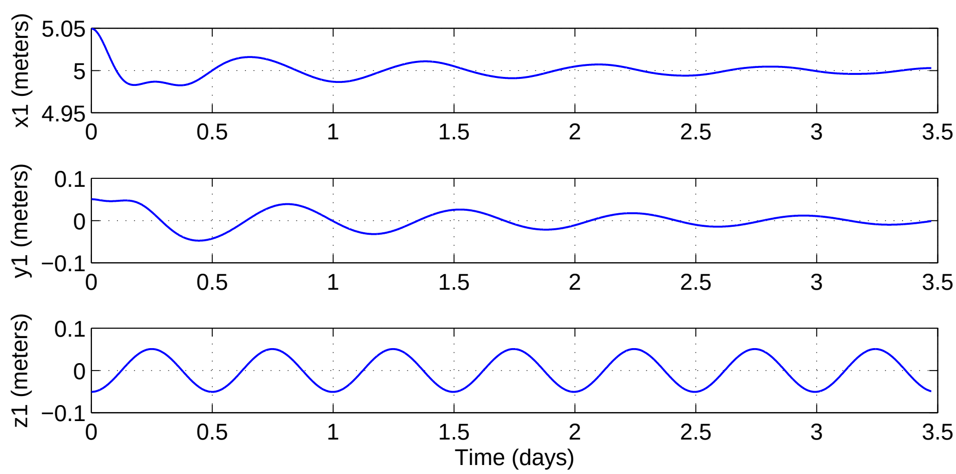
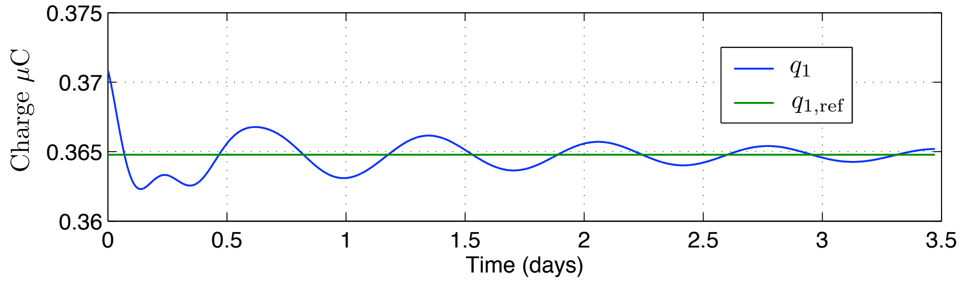
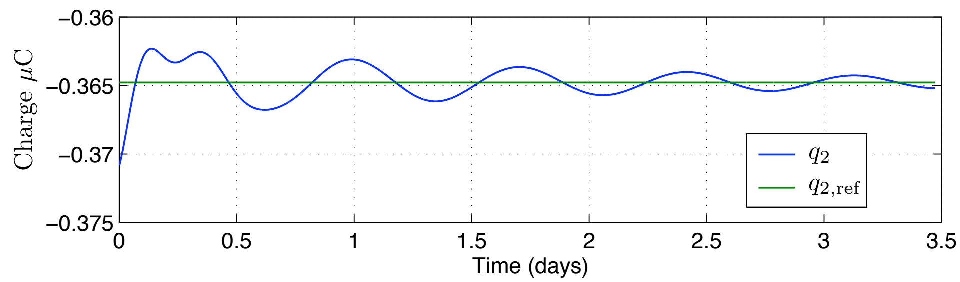


Figure 6: Convergence of x_1, y_1 and z_1 Simulated Using Cartesian Hill's Equations with Full Non-Linear Coulomb Force Term



(a) Spacecraft 1



(b) Spacecraft 2

Figure 7: Spacecraft Control Charge Illustrations.

$(\psi, \theta, \delta L)$ are used. With the presented charge feedback law, both the yaw motion ψ and the separation distance deviation δL converged to zero. The wiggle seen in the otherwise critically damped δL is due to the $\dot{\psi}$ coupling term which acts as a driving force. The damping of ψ occurs due to the phase difference between the driving $\delta \dot{L}$ term and ψ in Eq. (40b). For the set of initial conditions the restoration to equilibrium occurs in about 2 days or two orbit revolutions at GEO. As expected, the pitch motion $\theta(t)$ was a stable sinusoidal motion. The values of the x_1, y_1 and z_1 coordinates calculated from δL, ψ and θ values using Eq. (10) are plotted in Figure 5(b).

Figure 6 shows the solution of the nonlinear equation under these conditions. From the graph it is evident that the initial behavior is captured by the linearized model and this in essence verifies the linearization. However, the approach to convergence is slower in the nonlinear case.

Figure 7(a) and Figure 7(b) show the spacecraft control charges q_1 and q_2 for the case where the Cartesian Hill frame coordinates are integrated with the full nonlinear Coulomb force computation. The charges are equal in magnitude and opposite in polarity. Both are converging to their respective reference values pertaining to the static equilibrium. Note that the deviation from the value of reference charges is small, justifying the linearization assumptions used. The magnitude of the control charges is in the order of micro-Coulomb which is easily realizable in practice using charge emission devices.

CONCLUSION

The concept of a Coulomb (electrostatic) tether is introduced to bind two satellites in a near-rigid formation. While the Coulomb force cannot stabilize the attitude, the gravity gradient torque is exploited to stabilize the Coulomb tether formation about the orbit nadir direction. The analysis is based on a linearized model whose validity is also shown. It was observed that a simple feedback law for the restoring charge in terms of the change in relative separation and its rate is adequate for separation distance control. The control charge needed are small and realizable in practice.

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